MA4002 Final Exam Solutions 1997

1.(i) $v = 3 - 3e^{-2t}$; $\bar{v} = \frac{1}{2}(5 + e^{-6})$.

(ii) Substitute $u = \ln x$. Answer: $\ln|\ln x| + C$; $\ln(\ln\infty)$ and $\ln(\ln 0)$ don't exist.

(iii) $\tan 2$. **(iv)** Try it yourself! (Answer: 6.) **(v)** $S_n = \sum_{i=1}^{n} \frac{3}{n} \sin\left(\frac{3(i-1)}{n}\right)$.

(vi) $V = \int_{-1}^{1} \pi(1-x^2)^2\,dx = \frac{16\pi}{15}$. **(vii)** Integrate by parts with $u = x^{2n-1}$ and $dv = xe^{-x^2}\,dx$.

(viii) $e_Q = 2e_x + |y\cot y|e_y$. **(ix)** Variables separable. $1 + y^2 = x$, giving $y = \sqrt{x-1}$.

(x) 0, since $C_2 = aC_1$.

2.(i) Integrate by parts twice with $u = e^{2x}$ each time. Answer: $\frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x + C$.

(ii) Use long division and partial fractions to write integrand as $x + 1 + \frac{1}{x+2} + \frac{3}{x-2}$.
Answer: $\frac{x^2}{2} + x + \ln|x+2| + 3\ln|x-2| + C$.

(iii) Substitute $x = \sec\theta$ to obtain $\int_0^{\frac{\pi}{3}} \tan^2\theta\,d\theta = \int_0^{\frac{\pi}{3}} (\sec^2\theta - 1)\,d\theta = \sqrt{3} - \frac{\pi}{3}$.

3. (i) $A = \int_3^5 2\sqrt{25-x^2}\,dx = \frac{25\pi}{2} - 25\sin^{-1}\frac{3}{5} - 12$, after substituting $x = 5\sin\theta$.

(ii) By shells $V = \int_0^2 2\pi x(10 - x - x^3)\,dx = \frac{328}{15}\pi$. (iii) $s = \int_1^3 (x^2 + \frac{1}{4x^2})\,dx = \frac{53}{6}$.

(iv) $M = \int_0^{\frac{\pi}{2}} (2 + \sin x)\,dx = \pi + 1$. $I = \int_0^{\frac{\pi}{2}} x^2(2+\sin x)\,dx = \frac{\pi^3}{12} + \pi - 2$, integrating by parts twice.

4.(a) $h = \frac{\pi}{4}$. $S_4 \approx 6.19455$. $E_S < \frac{h^4\pi}{15} \approx 0.07969$. $E_S < \frac{\pi^5}{240n^4} < 10^{-10} \Rightarrow 2n > 672$.

(b) $x_n = 0.5n$. Euler's method: $y_{n+1} = 2y_n + 0.125n^2$, with $y_0 = 1$. $y(1) = y_2 = 4.125$.
Improved Euler's method: $y_{n+1} = 2.5y_n + 0.25n^2 + 0.125n + 0.0625$, with $y_0 = 1$. $y(1) = y_2 = 6.84375$.

5.(a) $f(1+h,k) = 1 + 2h + 2k + h^2 + 3hk + \frac{1}{2}k^2 + \cdots$.

(b) Fit line to points $(\ln x, \ln y)$; slope $\alpha \approx 1.934$ and intercept $\ln k \approx 1.172$, giving $k \approx 3.229$.

6.(a) Integrating factor is $e^{\frac{kt}{m}}$. Solution: $v = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{\frac{-kt}{m}}$. Terminal velocity is $\frac{mg}{k}$.
Time taken is $\frac{m\ln 10}{k}$.

(b) Char. eqn. $\lambda^2 + 5\lambda + 4 = 0 \Rightarrow \lambda = -1, -4$. So $y_h = Ae^{-x} + Be^{-4x}$. Try $y_p = \alpha x + \beta$, to find $\alpha = 1$, $\beta = -1$. Hence $y = Ae^{-x} + Be^{-4x} + x - 1$. Applying initial conditions gives $A = 4$, $B = -1$.

7.(a)

(i) RREF: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$

Infinite number of solutions: *viz.* $(-1-t, \frac{2}{3} - \frac{t}{3}, t)$, any $t \in \mathbf{R}$.

(ii) RREF: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

No solution (inconsistent).

(b) $A^{-1} = \begin{bmatrix} \frac{33}{2} & \frac{1}{2} & -5 \\ \frac{19}{2} & \frac{1}{2} & -3 \\ -3 & 0 & 1 \end{bmatrix}$.