MA4002 Final Exam Solutions 1998

1.(i) $V = \int_0^7 \frac{200}{(t+3)^3} = \frac{91}{9}$.

(ii) Use partial fractions: $\frac{x}{x^2-3x+2} = -\frac{1}{x-1} + \frac{2}{x-2}$. Answer: $-\ln|x-1| + 2\ln|x-2| + C$.

(iii) $3\cos(9x^2)$. **(iv)** Integrate by parts. Answer: $\frac{1}{2} - \frac{3}{2}e^{-2}$. **(v)** Try it yourself!

(vi) $\int_1^4 \sqrt{x}\,dx = \frac{14}{3}$. **(vii)** $f_x = y\sec^2 xy$; $f_y = x\sec^2 xy$;
$f_{xx} = 2y^2\sec^2 xy\tan xy$; $f_{xy} = \sec^2 xy + 2xy\sec^2 xy\tan xy$; $f_{yy} = 2x^2\sec^2 xy\tan xy$.

(viii) Variables separable: $y = \frac{1}{1-\ln|x|}$. **(ix)** $y_{n+1} = y_n + 0.2(0.2n + y_n)^2$, $y_0 = 1$. **(x)** -1.

2.(i) Substitute $u = 1 + 2\cos^2 t$. Answer: $\frac{1}{4}(\ln 3 - \ln 2)$. **(ii)** Substitute $u = x - 3$ to obtain $\int_{-3}^{-2}\left(1 + \frac{6u}{u^2+1} + \frac{8}{u^2+1}\right)du$. Answer: $1 - 3\ln 2 + 8\tan^1(-2) - 8\tan^{-1}(-3)$.

(iii) Integrate by parts with $u = (\ln x)^2$ and $dv = x^{\frac{1}{2}}\,dx$.
Then integrate by parts with $u = \ln x$ and $dv = x^{\frac{1}{2}}\,dx$. Answer: $x^{\frac{3}{2}}(\frac{2}{3}(\ln x)^2 - \frac{8}{9}\ln x + \frac{16}{27}) + C$.

3. (i) $A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\sqrt{2} - \sec x)\,dx = \frac{\pi}{\sqrt{2}} + 2\ln(\sqrt{2}-1)$. (ii) By washers, $V = \int_0^3 \pi(3y - y^2)^2\,dy = \frac{81}{10}\pi$.

(iii) $M = \int_0^4 \rho_0(2 + x^{\frac{1}{2}})\,dx = \frac{40\rho_0}{3}$. $I = \int_0^4 \rho_0(2 + x^{\frac{1}{2}})(x-1)^2\,dx = \frac{1224\rho_0}{35}$.

(iv) $|\mathbf{r}'(t)|^2 = 5e^{-2t}$, so $s = \int_0^1 \sqrt{5}e^{-t}\,dt = \sqrt{5}(1 - e^{-1})$.

4.(a) $\frac{\sqrt{\pi}}{4}$; integrate by parts with $u = x^{2n-1}$ and $dv = xe^{-x^2}\,dx$;
in last part use the fact that $(2n-1)\cdot(2n-3)\cdots 5\cdot 3\cdot 1 = \frac{(2n)!}{2^n n!}$.

(b) $f(1+h, 1+k) = \ln 2 + 1 + h + k + \frac{h^2}{2} - 2hk + \frac{k^2}{2} + \cdots$.

5.(a) $-4 = m = -\frac{1}{34}(a - 2b + 120)$ and $\frac{23}{5} = c = \frac{1}{5}(a + b + 37)$. Solve to get $a = -4$ and $b = -10$.

(b) Use $h = \frac{1}{4}$ and $y_i = \cos(e^{\frac{i}{4}})$ to get $S_4 \approx -0.12279$. $E_S < \frac{h^4}{3} \approx 0.0013$. $E_S < \frac{1}{24n^4} < 10^{-8} \Rightarrow 2n \geq 76$.

6.(a) Integrating factor is $e^{\frac{Rt}{L}}$. Solution: $i(t) = \frac{E_0}{R}(1 - e^{-\frac{Rt}{L}})$. As $t \to \infty$, $i(t) \to \frac{E_0}{R}$.

(b) Char. eqn. $\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = -1, -1$. So $y_h = Ae^{-x} + Bxe^{-x}$.
Try $y_p = \kappa x^2 e^{-x}$, to find $\kappa = 1$. Hence the general solution is $y = Ae^{-x} + Bxe^{-x} + x^2e^{-x}$.

7.(a)

(i) RREF: $\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{7}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 1 \end{array}\right]$
No solution (inconsistent).

(ii) RREF: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$
Unique solution $(x, y, z) = (3, -1, 0)$.

(b) $A^{-1} = \begin{bmatrix} -5 & 8 & -3 \\ -2 & 3 & -1 \\ 8 & -11 & 4 \end{bmatrix}$.