MA4002 Final Exam Solutions 1999

1.(i) $v = 100 - 100e^{-\frac{t}{10}}$; $\quad s = 100t + 1000e^{-\frac{t}{10}}$.

(ii) Integrate by parts with $u = \ln x$ and $dv = x^3\,dx$. Answer: $\frac{x^4}{4}\ln x - \frac{x^4}{16} + C$. **(iii)** $\tan(12)$.

(iv) $\frac{\pi}{n}\sum_{i=1}^{n}\cos\left(\frac{(2i-1)\pi}{2n}\right)$. **(v)** `with(student): leftsum(exp(x^2),x=0..3,3000);`

(vi) $V = \int_{-1}^{1}\pi(5^2 - (5x^2)^2)\,dx = 40\pi$. **(viii)** $e_Q = |2x\tan(2x+3y)|e_x + |1-3y\tan(2x+3y)|e_y$.

(vii) $y_{n+1} = y_n + 0.1(0.2n - y_n^3 + 0.2(n+1) - (y_n + 0.04n - 0.2y_n^3)^3)$, where $y_0 = 2$.

(ix) Linear, integrating factor $v = e^{2x}$. Solution: $y = e^{-x} + e^{-2x}$. **(x)** $\beta = 10$.

2.(i) Substitute $u = e^{3t} + 3e^{-t} + 6$. Answer: $\frac{1}{3}\ln|e^{3t} + 3e^{-t} + 6| + C$. **(ii)** Integrate by parts with $u = x$ and $dv = \sec^2 x\,dx$. Use log-table formula for $\int \tan x dx$ to obtain answer $\frac{\pi}{4} - \frac{1}{2}\ln 2$.

(iii) Substitute $u = \tan(\frac{t}{2})$ to obtain

$$\int_0^{\sqrt{2}-1}\frac{1}{(2u-1)(u-2)}\,du = \frac{1}{3}\int_0^{\sqrt{2}-1}\left(\frac{1}{u-2} - \frac{1}{u-\frac{1}{2}}\right)du = \frac{1}{3}\left(\ln(3-\sqrt{2}) - \ln(3-2\sqrt{2}) - \ln 2\right).$$

3. (i) $\int_0^\infty (2\cosh x - e^x)\,dx = \int_0^\infty e^{-x}\,dx = 1$. (ii) By cylindrical shells $V = 2\pi\int_0^2 (14x - 3x^2 - x^4)\,dx$ $= \frac{136\pi}{5}$. (iii) $s = \int_1^2 (\frac{y^2}{2} + \frac{1}{2y^2})\,dy = \frac{17}{12}$. (iv) $M = \int_0^1 \sqrt{4-x^2}\,dx = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$, (subs $x = 2\sin\theta$).

$\bar{x}M = \int_0^1 x\sqrt{4-x^2}\,dx = \frac{8}{3} - \sqrt{3}$, so $\bar{x} = \dfrac{8-3\sqrt{3}}{\pi + \frac{3\sqrt{3}}{2}}$.

4.(a) Integrate by parts with $u = \cos^{2n+2}x$ and $dv = \cos x\,dx$, and use $\sin^2 x = (1-\cos^2 x)$ in remaining integral to obtain $I_{n+1} = (2n+2)I_n - (2n+2)I_{n+1}$.

Hence $I_n = \frac{2n}{2n+1}I_{n-1} = \cdots = \frac{(2n)(2n-2)\cdots 6.4.2}{(2n+1)(2n-1)\cdots 7.5.3}I_0 = \frac{2^n n!}{\left(\frac{(2n+1)!}{2^n n!}\right)}.1 = \frac{2^{2n}\,(n!)^2}{(2n+1)!}$.

(b) $\sum x_i = 18$, $\sum y_i = 31$, $\sum x_i^2 = 88$, $\sum x_i y_i = 129$. So the least squares line is $y = \frac{3}{4}x + \frac{7}{2}$.

5.(a) $f(h, k+\pi) = \pi^2 + (\pi^2 - 3)h + (2\pi - 1)k + \frac{1}{2}\pi^2 h^2 + 2\pi hk + k^2 + \cdots$.

(b) $\left|\frac{d^2}{dx^2}\ln(x^2+1)\right| = \left|\frac{2-2x^2}{(x^2+1)^2}\right| \le 2$ for $x \in [0,1]$. $\quad h = \frac{1}{n}$. $E_T < \frac{1}{6n^2} < 10^{-10}$ if $n \ge 4083$.

6.(a) Characteristic equation $\lambda^2 + 2\lambda + 5 = 0 \Rightarrow \lambda = -1 \pm 2i$. So $q_h(t) = Ae^{-t}\cos 2t + Be^{-t}\sin 2t$.

(b) Try $q_p = \alpha\cos t + \beta\sin t$, to find $\alpha = -\frac{1}{2}$, $\beta = 1$. So $q(t) = Ae^{-t}\cos 2t + Be^{-t}\sin 2t - \frac{1}{2}\cos t + \sin t$.

(c) $q(t) = \frac{3}{2}e^{-t}\cos 2t + \frac{1}{4}e^{-t}\sin 2t - \frac{1}{2}\cos t + \sin t$.

7.(a) (i) $x+y=0$, $x+y=1$, $x+y=2$, $x+y=3$. (ii) $x=1$, $y=2$, $x+y=3$, $x-y=-1$.
(iii) $x+y=1$, $2x+2y=2$, $3x+3y=3$, $4x+4y=4$.

(b) $A^{-1} = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 2 & -3 & 1 \end{bmatrix}$. $\quad AA^{-1} = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $\quad \det A = 1$.