MA4002 Final Exam Solutions 2000

1.(a) $T = 30 + 50e^{-0.02t}$. As $t \to \infty$ $T \to 30$. **(b)** $\displaystyle\int_0^1 \frac{1}{1+x^2} = \frac{\pi}{4}$.

(c) Put $u = \ln x$. Then $y = \displaystyle\int_{-\infty}^u \frac{e^t}{t}\,dt$, so $\displaystyle\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{e^u}{u}\frac{1}{x} = \frac{1}{\ln x}$, using FTC I. **(d)** Do it!

(e) $\displaystyle\int_2^\infty \frac{x^5}{x^6-1}\,dx \geq \int_2^\infty \frac{1}{x}\,dx = \infty$, so the integral diverges.

(f) $\bar{y} = \displaystyle\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\cos x\,dx = 1 - \frac{2}{\pi}$, using I by P. **(g)** $A(x) = 3x^2 + 6x$; $V = \displaystyle\int_0^2 (3x^2+6x)dx = 20$.

(h) $f(x,y) = \dfrac{x}{y} + \dfrac{y}{x}$; $f_x = \dfrac{1}{y} - \dfrac{y}{x^2}$; $f_y = -\dfrac{x}{y^2} + \dfrac{1}{x}$; $f_{xx} = \dfrac{2y}{x^3}$; $f_{yy} = \dfrac{2x}{y^3}$; $f_{xy} = -\dfrac{1}{y^2} - \dfrac{1}{x^2}$.

(i) Variables separable. $\sin^{-1} y = x^2 + C$. Solution: $y = \sin(x^2 - 1)$. **(j)** 4.

2.(a) Substitute $u = \tan x$. Answer: $\sin^{-1}(\tan x) + C$.

(b) Integrate by parts twice with $u = e^{3x}$ both times to get $I = e^{3\pi} + 1 - 9I$, so $I = \frac{1}{10}(e^{3\pi} + 1)$.

(c) $I = \displaystyle\int (x - 2 + \frac{2x+20}{x^2+6x+10})\,dx$. Now substitute $u = x+3$ in rational function.

Answer: $\frac{x^2}{2} - 2x + \ln(x^2+6x+10) + 14\tan^{-1}(x+3) + C$.

3.(a) $A = \displaystyle\int_{-2}^3 (12 + 2x - 2x^2)\,dx = \frac{125}{3}$. (b) By cylindrical shells, $V = 2\pi \displaystyle\int_{b-a}^{b+a} x.2\sqrt{a^2 - (x-b)^2}\,dx$ $= 2\pi^2 a^2 b$, using substitution $u = x - b$. (c) $s = \displaystyle\int_0^2 t\sqrt{t^2+4} = (16\sqrt{2} - 8)/3$.

(d) $M = \displaystyle\int_0^8 (2\times 4)(2 + \frac{x}{4})\,dx = 192$; $\bar{x} = \displaystyle\frac{1}{M}\int_0^8 x(2\times 4)(2+\frac{x}{4})\,dx = \frac{40}{9}$.

4.(a) Integrate by parts with $u = (\ln t)^n$ and $dv = dt$. $I_5 = e - 5I_4 = \cdots = 76e - 120I_0 = 120 - 44e$.

(b) $y = y_0 e^{-t\ln 2/1000}$. Put $t = 700$ to get $y = .6155\,y_0$, so 61.55% remains.

(c) $y_{n+1} = y_n + 0.2(0.04n^2 + \sqrt{y_n})$. $y_0 = 1$. $y(0.6) \approx y_3 = 1.698$.

5.(a) $\ddot{x} + 4x = 0$. So $x = A\cos 2t + B\sin 2t$. Applying initial conditions gives $A = 2$ and $B = \frac{3}{2}$. Amplitude $= \sqrt{A^2+B^2} = \frac{5}{2}$.

(b) $\ddot{x} + 5\dot{x} + 4x = 0$. So $x = Ae^{-t} + Be^{-4t}$. Applying initial conditions gives $A = \frac{11}{3}$ and $B = -\frac{5}{3}$.

(c) $\ddot{x} + 5\dot{x} + 4x = 34\cos t$. Particular solution $x_p = \alpha\cos t + \beta\sin t$. Sub in to find $\alpha = 3$ and $\beta = 5$. So $x = x_h + x_p = Ae^{-t} + Be^{-4t} + 3\cos t + 5\sin t$.

6.(a) $h = \dfrac{1}{n}$ and $M_4 < 2$, so $E_S < \dfrac{h^4}{180}(b-a)M_4 < \dfrac{1}{45n^4} < 10^{-6}$ if $n > 12.2$. So 26 intervals suffice.

(b) $\frac{\Delta w}{w} \approx 2\frac{\Delta x}{x} + 3\frac{\Delta y}{y} - 2\frac{\Delta z}{z} = 2(-.04) + 3(.02) - 2(-.05) = .08$. So w increases by 8% (approx.).

(c) $f(h,k) = 1 - k - \frac{1}{2}h^2 + k^2 + \cdots$.

7.(a)(i) Subtracting equation 3 from equation 2 gives $2y + 2z = -1$, which contradicts equation 1. So no solution. (ii) $(x,y,z) = (0,-2,3)$, unique. **(b)** $A^{-1} = \begin{bmatrix} -8 & -7 & 6 \\ 7 & 6 & -5 \\ 12 & 11 & -9 \end{bmatrix}$.