MA4002 Final Exam Solutions 2001

1.(a) $v = 20t + 20t^{\frac{3}{2}}$. $\quad s = 10t^2 + 8t^{\frac{5}{2}}$. $\quad s(4) = 416$.

(b) $\frac{1}{2}(0^2 + 0.5^2 + 1^2 + 1.5^2) = 1.75$.

(c) Near $x = 0$ integral behaves like $\displaystyle\int \frac{2}{2x}\,dx$ which is like $\ln x$ near $x = 0$ and so is divergent.

(d) Integrate by parts with $u = x^n$ and $dv = e^x\,dx$.

(e) $f_x = -2x\sin(x^2+y)$; $\quad f_y = -\sin(x^2+y)$;

$f_{xx} = -2\sin(x^2+y) - 4x^2\cos(x^2+y)$; $\quad f_{yy} = -\cos(x^2+y)$; $\quad f_{xy} = -2x\cos(x^2+y)$.

(f) $x_n = 0.2n$. $\quad y_{n+1} = y_n + 0.1(0.2n + \ln(y_n) + 0.2(n+1) + \ln(y^\star_{n+1}))$,

where $y^\star_{n+1} = y_n + 0.2(0.2n + \ln(y_n))$, starting with $y_0 = 1$.

(g) Integrating factor $v = x^2$, so $(x^2y)' = 3x^2$. Integrating gives $x^2y = x^3 + C$ and so $y = x + \dfrac{C}{x^2}$

(h) Put determinant of coefficients equal to zero to get $\beta = 9$.

2. (a) $A = \displaystyle\int_0^\infty \frac{1}{4(x^2 + (\frac{3}{2})^2)}\,dx = \frac{1}{4}\left(\frac{2}{3}\tan^{-1}\frac{2x}{3}\right)\Big|_0^\infty = \frac{1}{6}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{12}$.

(b) By cylindrical shells, $V = 2\pi\displaystyle\int_0^1 y(5y^3 + 3y)\,dy = 4\pi$.

(c) $s = \displaystyle\int_1^2 \sqrt{1 + (y')^2}\,dx = \int_1^2 \sqrt{1 + (\frac{x}{4} - \frac{1}{x})^2}\,dx = \int_1^2 \left(\frac{x}{4} + \frac{1}{x}\right)dx = \frac{3}{8} + \ln 2$.

(d) $M = \displaystyle\int_0^{\frac{\pi}{2}} (1 + \cos x)\,dx = 1 + \frac{\pi}{2}$; $\quad \bar{x} = \displaystyle\frac{1}{M}\int_0^{\frac{\pi}{2}} x(1+\cos x)\,dx = \frac{\pi^2/8 + \pi/2 - 1}{1 + \pi/2} \approx 0.7019$.

3.(a) $\ddot{q_h} + 6\dot{q_h} + 10q_h = 0$ has characteristic equation $\lambda^2 + 6\lambda + 10 = 0$, which has solutions $\lambda = -3 \pm i$. This leads to general solution $q_h = Ae^{-3t}\cos t + Be^{-3t}\sin t$.

(b) Put $q_p = \alpha\cos t + \beta\sin t$. Then $39\cos t = \ddot{q_p} + 6\dot{q_p} + 10q_p = (9\alpha + 6\beta)\cos t + (9\beta - 6\alpha)\sin t$. So we get the simultaneous equations $9\alpha + 6\beta = 39$ and $9\beta - 6\alpha = 0$ which have solutions $\alpha = 3$ and $\beta = 2$. Hence the general solution is $q = q_h + q_p = Ae^{-3t}\cos t + Be^{-3t}\sin t + 3\cos t + 2\sin t$.

(c) Applying the initial conditions to q and $\frac{dq}{dt}$, we find $A = 1$ and $B = 1$ and so

$q = e^{-3t}\cos t + e^{-3t}\sin t + 3\cos t + 2\sin t$.

4.(a) $z_x = \dfrac{2x}{x^2 + 2y^2}$ $\quad z_y = \dfrac{4y}{x^2+2y^2}$ $\quad z_{xx} = \dfrac{4y^2 - 2x^2}{(x^2+2y^2)^2}$ $\quad z_{yy} = \dfrac{4x^2 - 8y^2}{(x^2+2y^2)^2}$ $\quad z_{xy} = \dfrac{-8xy}{(x^2+2y^2)^2}$.

$z(1,1) = \ln 3$; $\quad z_x(1,1) = \frac{2}{3}$; $\quad z_y(1,1) = \frac{4}{3}$; $\quad z_{xx}(1,1) = \frac{2}{9}$; $\quad z_{yy}(1,1) = -\frac{4}{9}$; $\quad z_{xy}(1,1) = -\frac{8}{9}$.

So $f(1+h, 1+k) = \ln 3 + \frac{2h}{3} + \frac{4k}{3} + \frac{h^2}{9} - \frac{8hk}{9} - \frac{2k^2}{9} +$ higher order terms.

(b) $\sum x = 14$, $\quad \sum y = 16 + a + b$, $\quad \sum x^2 = 62$, $\quad \sum xy = a + 3b + 8$.

$m = -3$ then gives $9a - b = 158$ and $c = 16$ gives $a + b = 22$. Solving these gives $a = 18$ and $b = 4$.

5.(a) $\begin{pmatrix} 2 & -2 & 4 & -4 \\ 6 & -3 & 8 & -6 \\ 12 & 0 & 8 & 0 \end{pmatrix}$ $\qquad$ **(b)** -9. $\qquad$ **(c)** $\begin{pmatrix} -13 & -16 & 7 \\ 6 & 7 & -3 \\ 4 & 5 & -2 \end{pmatrix}$.