MA4002 Final Exam Solutions 2003

1.(a) $v = 20\sin(0.5t)$. $s = 40 - 40\cos(0.5t)$. $s(3) \approx 37.17m$. **(b)** $e^2 + e^3 \approx 27.47$.

(c) $u = \cos^{2n} x$ and $dv = \cos x\, dx$, so $du = -2n\cos^{2n-1} x \sin x\, dx$ and $v = \sin x$. Replace $\sin^2 x$ by $1 - \cos^2 x$ to get $I_n = 2nI_{n-1} - 2nI_n$. **(d)** $V = \pi \int_{-2}^{2} (2 - x^2)^2\, dx = \frac{512\pi}{15}$.

(e) $f_x = y\cos(xy)$; $f_y = x\cos(xy)$;
$f_{xx} = -y^2 \sin(xy)$; $f_{yy} = -x^2 \sin(xy)$; $f_{xy} = \cos(xy) - xy\sin(xy)$.

(f) $x_n = 0.1n$. $y_{n+1} = y_n + 0.05(\cos(0.1n + 2y_n) + \cos(0.1(n+1) + 2y^\star_{n+1}))$,
where $y^\star_{n+1} = y_n + 0.1(\cos(0.1n + 2y_n))$, starting with $y_0 = 2$.

(g) Integrating factor $v = \frac{1}{x^3}$, so $\left(\frac{y}{x^3}\right)' = \frac{1}{x}$. Integrating and multiplying by x^3 gives $y = Cx^3 + x^3 \ln x$.

(h) -61.

2. (a) By cylindrical shells, $V = \int_0^1 2\pi x e^{-x^2}\, dx = \pi - \frac{\pi}{e}$.

(b) $A = \int_1^{e^2} \ln x\, dx = e^2 + 1$ after using integration by parts with $u = \ln x$ and $dv = dx$.

(c) $s = \int_1^3 \sqrt{1 + (y')^2}\, dx = \int_1^3 \sqrt{1 + (x^2 - \frac{1}{4x^2})^2}\, dx = \int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx = \frac{53}{6}$.

(d) $M = \int_0^2 \frac{1}{\sqrt{16 - x^2}}\, dx = \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$; $\bar{x} = \frac{1}{M}\int_0^2 \frac{x}{\sqrt{16 - x^2}}\, dx = \frac{6}{\pi}(4 - \sqrt{12}) \approx 1.023$.

3.(a) $\ddot{q_h} + 4\dot{q_h} + 4q_h = 0$ has auxiliary equation $\lambda^2 + 4\lambda + 4 = 0$, which has solution $\lambda = -2$. This leads to general solution $q_h = Ae^{-2t} + Bte^{-2t}$.
(b) Put $q_p = \alpha\cos 2t + \beta\sin 2t$. Then $\cos 2t + \sin 2t = \ddot{q_p} + 4\dot{q_p} + 4q_p = 8\beta\cos 2t - 8\alpha\sin 2t$. So we get $\alpha = -\frac{1}{8}$ and $\beta = \frac{1}{8}$. Hence the general solution is $q = q_h + q_p = Ae^{-2t} + Bte^{-2t} - \frac{1}{8}\cos 2t + \frac{1}{8}\sin 2t$.
(c) Applying the initial conditions to q and $\frac{dq}{dt}$, we find $A = \frac{9}{8}$ and $B = 2$ and so
$q = \frac{9}{8}e^{-2t} + 2te^{-2t} - \frac{1}{8}\cos 2t + \frac{1}{8}\sin 2t$.

4.(a) $z_x = \frac{2x}{x^2 + 3y}$ $z_y = \frac{3}{x^2 + 3y}$ $z_{xx} = \frac{6y - 2x^2}{(x^2 + 3y)^2}$ $z_{yy} = \frac{-9}{(x^2 + 3y)^2}$ $z_{xy} = \frac{-6x}{(x^2 + 3y)^2}$.
$z(1,1) = \ln 4$; $z_x(1,1) = \frac{1}{2}$; $z_y(1,1) = \frac{3}{4}$; $z_{xx}(1,1) = \frac{1}{4}$; $z_{yy}(1,1) = -\frac{9}{16}$; $z_{xy}(1,1) = -\frac{3}{8}$.
So $f(1 + h, 1 + k) = \ln 4 + \frac{h}{2} + \frac{3k}{4} + \frac{h^2}{8} - \frac{3hk}{8} - \frac{9k^2}{32} +$ higher order terms.

(b) $\sum x = 0$, $\sum y = 16 + a + b$, $\sum x^2 = 10$, $\sum xy = a + 2b - 24$.
$m = -\frac{17}{5}$ then gives $a + 2b = -10$ and $c = 2$ gives $a + b = -6$. Solving these gives $a = -2$ and $b = -4$.

5.(a) (i) Inconsistent -no solution. (ii) An infinite number of solutions: $(x, y, z) = (3 - t, -2 - t, t)$ is a solution for any real number t.

(b) $$\begin{pmatrix} -2 & -2 & 1 \\ 9 & 10 & -3 \\ 3 & 3 & -1 \end{pmatrix}.$$