MA4002 Final Exam Answers, Spring 2005

1.(a) $v = 40 - 10e^{-0.1t}$. $\quad s = (40t + 100e^{-0.1t})\Big|_0^{20} = 700 + 100e^{-2} \approx 713.53$.

(b) $4 \cdot 3^{1/4}$.

(c) The cross-sectional area: $\pi(e^{-x})^2 = \pi e^{-2x}$. $\quad V = \pi \int_0^4 e^{-2x}\,dx = \frac{\pi}{2}(1 - e^{-8}) \approx 1.57$.

(d) Reduction formula: $I_n = x(\ln x)^n|_1^{e^2} - nI_{n-1} = 2^n e^2 - nI_{n-1}$.

$I_0 = \int_1^{e^2} 1\,dx = e^2 - 1$; $\quad I_1 = 2e^2 - I_0 = e^2 + 1$; $\quad I_2 = 4e^2 - 2I_1 = 2e^2 - 2$; $\quad I_3 = 8e^2 - 3I_2 = 2e^2 + 6$.

(e) $f_x = y\cos(xy)$, $f_y = x\cos(xy)$, $f_{xx} = -y^2\sin(xy)$, $f_{yy} = -x^2\sin(xy)$, $f_{xy} = \cos(xy) - xy\sin(xy)$.

(f) $x_n = 0.2n$. Start with $y_0 = 1$. $\quad y_{n+1} = y_n - 0.1(x_n y_n + x_{n+1} y^\star_{n+1})$, where $y^\star_{n+1} = y_n - 0.2x_n y_n$.

$y^\star_1 = 1 - 0.2(0 \times 1) = 1$, $\quad y(0.2) \approx y_1 = 1 - 0.1(0 \times 1 + 0.2 \times 1) = 0.98$.

$y^\star_2 = 0.98 - 0.2(0.2 \times 0.98) = 0.9408$, $\quad y(0.4) \approx y_2 = 0.98 - 0.1(0.2 \times 0.98 + 0.4 \times 0.9408) = 0.922768$.

(g) Integrating factor: $\sigma = \exp\{\int \frac{5}{x} dx\} = x^5$. Then $(x^5 y)' = 5x^5$ and $y = \frac{5x}{6} + \frac{C}{x^5}$. By $y(1) = 1$ we have $C = \frac{1}{6}$ and $y = \frac{5x}{6} + \frac{1}{6x^5}$.

(h) 31.

2.(a) Area: $\quad \int_0^2 2x\ln(1+x^2)\,dx = (1+x^2)\ln(1+x^2) - (1+x^2)\Big|_0^2 = 5\ln 5 - 4 \approx 4.047$.

(b) Cylindrical shell area: $2\pi x e^{-x}$. $\;V = 2\pi\int_0^8 x e^{-x}\,dx = -2\pi(1+x)e^{-x}\Big|_0^8 = 2\pi(1 - 9e^{-8}) \approx 6.264$.

(c) $y'(x) = -\dfrac{2x}{1-x^2}$. $\quad \sqrt{1 + y'^2} = \dfrac{1+x^2}{1-x^2}$.

Arc-length: $\quad s = \int_0^{1/2}\left[-1 + \dfrac{2}{1-x^2}\right]dx = -x + \ln(1+x) - \ln(1-x)\Big|_0^{1/2} = -\dfrac{1}{2} + \ln 3 \approx 0.5986$.

(d) $\rho = \dfrac{1}{5(3-x)} + \dfrac{1}{5(x+2)}$, while $\; x\rho = \dfrac{3}{5(3-x)} - \dfrac{2}{5(x+2)}$. Center of mass: $\;\bar{x} = M/m = 0.5$.

Mass: $m = \int_0^1 \rho\,dx = (2/5)(\ln 3 - \ln 2) \approx 0.162$. Moment: $M = \int_0^1 x\rho\,dx = (1/5)(\ln 3 - \ln 2) \approx 0.081$.

3.(a) **(i)** $\; y = C_1 e^x + C_2 e^{-6x}$. $\quad$ **(ii)** $\; y = e^{-x}(C_1\sin x + C_2\cos x)$.

(b) Particular solution: $\; y_p = (-12x - 1)e^{-2x}$. General solution: $\; y = (-12x-1)e^{-2x} + C_1 e^x + C_2 e^{-6x}$.

(c) $\; y = (-12x - 1)e^{-2x} + 4e^x - e^{-6x}$.

4.(a) Answer: $f(x,y) \approx 2 + 3(x-1) + 3(y-1) + 2(x-1)^2 + 6(x-1)(y-1) + 2(y-1)^2$.

$f_x = [1 + (x+y)\,y\,]e^{xy-1}$, $\quad f_{xx} = [2y + (x+y)y^2]e^{xy-1}$,

$f_y = [1 + (x+y)\,x\,]e^{xy-1}$, $\quad f_{xy} = [2x + 2y + (x+y)xy\,]e^{xy-1}$, $\quad f_{yy} = [2x + (x+y)x^2]e^{xy-1}$.

(b) $\; n = 5$, $\;\sum_{k=1}^5 x_k = 10$, $\;\sum_{k=1}^5 x_k^2 = 50$, $\;\sum_{k=1}^5 y_k = 8$, $\;\sum_{k=1}^5 x_k y_k = 28$.

$a = \dfrac{n \cdot 28 - 10 \cdot 8}{n \cdot 50 - 10^2} = 0.4$, $\quad b = \dfrac{8 - a \cdot 10}{n} = 0.8$. $\quad$ Answer: $\; y = 0.4x + 0.8$.

5.(a) $AA^T = \begin{bmatrix} 25 & -3 \\ -3 & 23 \end{bmatrix}$.

(b) **(i)** $x = [1,\, -4,\, 9]^T$. $\quad$ **(ii)** From $\left[\begin{array}{ccc|c} 1 & 0 & 13 & 118 \\ 0 & 1 & 4 & 32 \end{array}\right]$ obtain $\; x = [118 - 13t,\, 32 - 4t,\, t]^T$.

(c) $A^{-1} = \begin{bmatrix} 78 & -31 & -8 \\ 97 & -39 & -10 \\ 10 & -4 & -1 \end{bmatrix}$.