MA4002 Final Exam Answers, Spring 2006

1.(a) The race consists of two stages: $t_1 = 2$, $v_1(t) = \int_0^t 4\,dt = 4t$ for $t \in [0,2]$ and $v_2 = v_1(2) = 8$. Now $s_1 = \int_0^2 v_1(t)\,dt = 2t^2|_0^2 = 8$, hence $s_2 = 100 - s_1 = 92$ and $t_2 = s_2/v_2 = 92/8 = 11.5$. Finally $t = t_1 + t_2 = 2 + 11.5 = 13.5$ seconds.

(b) The cross-sectional area: $\dfrac{\pi}{(x+1)^2}$. $\quad V = \pi \displaystyle\int_0^9 \frac{dx}{(x+1)^2} = \frac{9\pi}{10} \approx 2.8274.$

(c) Reduction formula: $\boxed{I_n = -n\pi^{n-1} - n(n-1)I_{n-2}.}$ $I_0 = \displaystyle\int_0^\pi \cos x\,dx = 0$; $I_2 = -2\cdot\pi^1 - 2\cdot 1\cdot I_0 = -2\pi$; $\quad I_4 = -4\cdot\pi^3 - 4\cdot 3\cdot I_2 = -4\pi^3 + 24\pi$; $\quad I_6 = -6\cdot\pi^5 - 6\cdot 5\cdot I_4 = -6\pi^5 + 120\pi^3 - 720\pi$.

(d) $f_x = [1 + 2x^2y]e^{x^2y}$, $f_y = x^3\,e^{x^2y}$, $f_{xx} = [6xy + 4x^3y^2]e^{x^2y}$, $f_{yy} = x^5\,e^{x^2y}$, $f_{xy} = [3x^2 + 2x^4y]e^{x^2y}$.

(e) $x_n = 0.2n$. Start with $y_0 = 1$. $\quad y_{n+1} = y_n - 0.1(x_n y_n^2 + x_{n+1}[y^\star_{n+1}]^2)$, where $y^\star_{n+1} = y_n - 0.2x_n y_n^2$. $y_1^\star = 1 - 0.2(0\times 1^2) = 1$, $\quad y(0.2) \approx y_1 = 1 - 0.1(0\times 1^2 + 0.2\times 1^2) = 0.98$. $\quad y_2^\star = 0.98 - 0.2(0.2\times 0.98^2) = 0.941584$, $\quad y(0.4) \approx y_2 = 0.98 - 0.1(0.2\times 0.98^2 + 0.4\times 0.941584^2) = 0.92532878$.

(f) Integrating factor: $\sigma = \exp\{\int 2x\,dx\} = e^{x^2}$. Then $(e^{x^2}y)' = e^x$ and $y = e^{-x^2}[e^x + C]$. By $y(0) = 0$ we have $C = -1$ and $y = e^{x-x^2} - e^{-x^2}$. $\quad$ **(g)** 62.

(h) By the Extreme-Value Theorem, $\exists\ A, B \in [a,b]\ :\ f(A) = \min\limits_{[a,b]} f$ and $f(B) = \max\limits_{[a,b]} f$. Furthermore, we have $f(A) \le \bar f \le f(B)$, where $\bar f = (b-a)^{-1}\int_a^b f(x)\,dx$. Finally, by the Intermediate-Value Theorem, $\exists\ c$ between A and B such that $f(c) = \bar f$.

2.(a) Cylindrical shell area: $2\pi x[(x+1)\sin x]$. $\quad V = 2\pi \displaystyle\int_0^\pi x(x+1)\sin x\,dx =$

$$= 2\pi(-x^2\cos x + 2\cos x + 2x\sin x + \sin x - x\cos x)\Big|_0^\pi = 2\pi(\pi^2 + \pi - 4) \approx 56.619.$$

(b) $y'(x) = \dfrac{x^2}{8} - \dfrac{2}{x^2}$; $\sqrt{1 + y'^2} = \dfrac{16 + x^4}{8x^2}$. Arc-length: $s = \displaystyle\int_2^4 \Big[\frac{2}{x^2} + \frac{x^2}{8}\Big]\,dx = \frac{x^3}{24} - \frac{2}{x}\Big|_2^4 = \frac{17}{6} \approx 2.833.$

(c) $\rho = \dfrac{x}{x^2+4}$, while $\quad x\rho = 1 - \dfrac{4}{x^2+4}$. Center of mass: $\quad \bar x = M/m = \dfrac{2(4-\pi)}{2\ln 2} \approx 1.2384.$

Mass: $m = \displaystyle\int_0^2 \rho\,dx = \frac{\ln 2}{2} \approx 0.34657$. Moment: $M = \displaystyle\int_0^2 x\rho\,dx = 2 - \frac{\pi}{2} \approx 0.4292.$

3.(a) **(i)** $\quad y = C_1e^{2x} + C_2e^{4x}$. $\quad$ **(ii)** $\quad y = e^{3x}(C_1\sin x + C_2\cos x)$.

(b) Particular solution: $\; y_p = (-x^2 - x)e^{2x}$. General solution: $\; y = (-x^2 - x)e^{2x} + C_1e^{2x} + C_2e^{4x}$.

(c) $\; y = (-x^2 - x)e^{2x} + 10e^{2x} - 4e^{4x}$.

4.(a) Answer: $\; f(x,y) \approx 1 + y - \frac{1}{2}y^2 + (x-1)y$.

$f_x = y\cos(xy) - y(1+xy)\sin(xy)$, $\qquad f_{xx} = -2y^2\sin(xy) - (1+xy)y^2\cos(xy)$,

$f_y = x\cos(xy) - x(1+xy)\sin(xy)$, $\qquad f_{yy} = -2x^2\sin(xy) - (1+xy)x^2\cos(xy)$,

$f_{xy} = -2xy\sin(xy) + \cos(xy) - (1+xy)xy\cos(xy) - (1+xy)\sin(xy)$.

(b) $\; n = 6$, $\; \displaystyle\sum_{k=1}^6 x_k = 0$, $\; \displaystyle\sum_{k=1}^6 x_k^2 = 34$, $\; \displaystyle\sum_{k=1}^6 y_k = 3$, $\; \displaystyle\sum_{k=1}^6 x_ky_k = 34$.

$a = \dfrac{n\cdot 34 - 0\cdot 3}{n\cdot 34 - 0^2} = 1$, $\quad b = \dfrac{3 - a\cdot 0}{n} = \dfrac{1}{2}$. $\quad$ Answer: $\; y = x + \frac{1}{2}$.

5.(a) **(i)** $x = [4,\ -2,\ 5]^T$. $\qquad$ **(ii)** From $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 14 \\ 0 & 1 & 2 & 8 \end{array}\right]$ obtain $\quad x = [14 - 2t,\ 8 - 2t,\ t]^T$.

(iii) From $\left[\begin{array}{ccc|c} 1 & -3 & -4 & -10 \end{array}\right]$ obtain $\quad x = [-10 + 3t_1 + 4t_2,\ t_1,\ t_2]^T$.

(b) $A^{-1} = \begin{bmatrix} -7 & -4 & 1 \\ 2 & 1 & 0 \\ -\frac{14}{3} & -\frac{8}{3} & \frac{1}{3} \end{bmatrix}$.