MA4002 Final Exam Answers, Spring 2007

1.(a) Velocity: $v(t) = 1000 - \int_0^t 3000e^{-2s}ds = -500 + 1500e^{-2t}$. The particle stops at time T such that $v(T) = 0$; hence $T = \frac{1}{2}\ln 3 \approx 0.5493$ seconds. Distance $s = \int_0^T v(t)dt = 500 - 250\ln 3 \approx 225.3469$ m.

(b) The cross-sectional area: $\pi(\sin x)^2$.

$$V = \pi \int_0^\pi (\sin x)^2 dx = \pi \int_0^\pi \frac{1-\cos(2x)}{2} dx = \pi\left(\frac{x}{2} - \frac{1}{4}\sin(2x)\right)\Big|_0^\pi = \frac{\pi^2}{2} \approx 4.9348.$$

(c) Reduction formula: $\boxed{I_n = e - nI_{n-1}.}$

$I_0 = e - 1$; $I_1 = e - 1 \cdot I_0 = 1$; $I_2 = e - 2 \cdot I_1 = e - 2$; $I_3 = e - 3 \cdot I_2 = 6 - 2e$.

(d) $f_x = y^2\cos(xy)$, $f_y = \sin(xy) + xy\cos(xy)$, $f_{xx} = -y^3\sin(xy)$, $f_{yy} = 2x\cos(xy) - x^2y\sin(xy)$, $f_{xy} = 2y\cos(xy) - xy^2\sin(xy)$.

(e) $x_n = 0.1n$. Start with $y_0 = 0$. $y_{n+1} = y_n + 0.05(\cos(x_n + y_n) + \cos(x_{n+1} + y^\star_{n+1}))$, where $y^\star_{n+1} = y_n + 0.1\cos(x_n + y_n)$. Now $y^\star_1 = 0 + .1\cos 0 = .1$; $y(0.1) \approx y_1 = 0 + .05(\cos(0) + \cos(.1 + .1)) \approx .099$. $y^\star_2 = .099 + 0.1\cos(.1 + .099) \approx .197$, $y(0.2) \approx y_2 = .099 + .05(\cos(.1 + .099) + \cos(.2 + .197)) \approx .1941$.

(f) By separating variables, get $y = \sqrt[3]{x^2 + \sin x + C}$. By $y(0) = 1$ we have $C = 1$ and $y = \sqrt[3]{x^2 + \sin x + 1}$. **(g)** 39.

(h) One example: let $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

2.(a) Cylindrical shell area: $2\pi x[\frac{1}{x+1} - \frac{1}{x+2}]$. $V = 2\pi \int_0^1 x[\frac{1}{x+1} - \frac{1}{x+2}]dx =$

$$= 2\pi(-\ln(x+1) + 2\ln(x+2))\Big|_0^1 = 2\pi(-3\ln 2 + 2\ln 3) \approx 0.74005.$$

(b) $y'(x) = \dfrac{-\sqrt{4 - x^{2/3}}}{x^{1/3}}$; $\sqrt{1 + y'^2} = \dfrac{2}{x^{1/3}}$. Arc-length: $s = \displaystyle\int_1^8 \frac{2}{x^{1/3}}dx = 3x^{2/3}\Big|_1^8 = 9$.

(c) $\rho = \ln x$, while $x\rho = x\ln x$. Center of mass: $\bar{x} = M/m = \dfrac{e^2+1}{4} \approx 2.097$.

Mass: $m = \displaystyle\int_1^e \rho\, dx = (x\ln x - x)\Big|_1^e = 1$. Moment: $M = \displaystyle\int_1^e x\rho\, dx = \left(\frac{x^2\ln x}{2} - \frac{x^2}{4}\right)\Big|_1^e = \frac{e^2+1}{4} \approx 2.097$.

3.(a) (i) $y = C_1e^{3x} + C_2e^{4x}$. **(ii)** $y = e^{2x}(C_1\sin x + C_2\cos x)$.

(b) Particular solution: $y_p = 11\sin x + 7\cos x$.

General solution: $y = 11\sin x + 7\cos x + C_1e^{3x} + C_2e^{4x}$. **(c)** $y = 11\sin x + 7\cos x - 10e^{3x} + 5e^{4x}$.

4.(a) Answer: $f(x,y) \approx 2(x-1) + 2(y-1) + (x-1)^2 + 3(x-1)(y-1) = -y - 3x + x^2 + 3xy$.

$f_x = 2x\ln(xy) + x + y/x$, $f_{xx} = 2\ln(xy) + 3 - y/x^2$,

$f_y = \ln(xy) + x^2/y + 1$, $f_{yy} = 1/y - x^2/y^2$,

$f_{xy} = 2x/y + 1/x$.

(b) $n = 6$, $\displaystyle\sum_{k=1}^6 x_k = 15$, $\displaystyle\sum_{k=1}^6 x_k^2 = 55$, $\displaystyle\sum_{k=1}^6 y_k = 19$, $\displaystyle\sum_{k=1}^6 x_ky_k = 65$.

$a = \dfrac{n\cdot 65 - 15\cdot 19}{n\cdot 55 - 15^2} = 1$, $b = \dfrac{19 - a\cdot 15}{n} = \dfrac{2}{3}$. Answer: $y = x + \frac{2}{3}$.

5.(a) (i) $x = [3,\ 7,\ 2]^T$. **(ii)** From $\left[\begin{array}{ccc|c} 1 & 0 & 23/3 & 55/3 \\ 0 & 1 & -9 & -11 \end{array}\right]$ obtain $x = [55/3 - (23/3)t,\ -11 + 9t,\ t]^T$.

(iii) From $\left[\begin{array}{ccc|c} 1 & 2/3 & 5/3 & 11 \end{array}\right]$ obtain $x = [11 - (2/3)t_1 - (5/3)t_2, t_1, t_2]^T$.

(b) $A^{-1} = \begin{bmatrix} -19/2 & 16 & 11/2 \\ -12 & 20 & 7 \\ 2 & -3 & -1 \end{bmatrix}$.