MA4002 Final Exam Answers, Spring 2008

1.(a) Velocity: $v(t) = 50 + \int_0^t \sin(0.1s)\,ds = 60 - 10\cos(0.1t)$. Distance $s(T) = \int_0^T v(t)dt = 60T - 100\sin(0.1T)$ m and $s(40) = \boxed{2400 - 100\sin(4) \approx 2475.68\,\text{m}}$.

(b) The cross-sectional area: $\pi(1/\sqrt{x^2-1})^2$.

$$V = \pi\int_2^4 \frac{dx}{x^2-1} = \frac{\pi}{2}\int_0^4 \Big(\frac{1}{x-1} - \frac{1}{x+1}\Big)dx = \frac{\pi}{2}\ln\frac{x-1}{x+1}\Big|_2^4 = \frac{\pi}{2}(2\ln 3 - \ln 5) \approx 0.92329.$$

(c) Integrating by parts yields $I_n = (n-1)\int_0^{\pi/2} \sin^{n-2} x \cos^2 x\,dx$. Invoking $\cos^2 x = 1 - \sin^2 x$, we get the desired reduction formula. Next, $I_1 = 1$ implies $I_3 = \frac{2}{3}$ and $I_5 = \frac{8}{15}$.

(d) $f_x = [1-xy]e^{-xy}$, $f_y = -x^2 e^{-xy}$, $f_{xx} = [-2y + xy^2]e^{-xy}$, $f_{yy} = x^3 e^{-xy}$, $f_{xy} = [-2x + x^2 y]e^{-xy}$.

(e) $x_n = 0.1n$. Start with $y_0 = 2$. $y_{n+1} = y_n + 0.05[(x_n - y_n^2) + (x_{n+1} - [y^\star_{n+1}]^2)]$, where $y^\star_{n+1} = y_n + 0.1(x_n - y_n^2)$. Now $y_1^\star = 2 + .1(0 - 2^2) = 1.6$; $y(0.1) \approx y_1 = 2 + .05[(0-2^2) + (0.1 - 1.6^2)] = 1.677$. $y_2^\star = 1.4057671$, $y(0.2) \approx y_2 \approx 1.45257$. $y_3^\star \approx 1.261577$, $y(0.3) \approx y_3 \approx 1.292497$.

(f) Integrating factor: $v = \exp\{\int \frac{2}{x}dx\} = x^2$. Then $(x^2 y)' = 4x^3$ and therefore $y = x^{-2}[x^4 + C] = x^2 + \frac{C}{x^2}$. By $y(1) = 2$ we have $C = 1$ and $\boxed{y = x^2 + \frac{1}{x^2}}$. **(g)** -23.

(h) For $x > 0$ we have $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$, while for $x < 0$ we have $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-x)' = \frac{1}{x}$. Therefore $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all $x \neq 0$. The desired result follows.

2.(a) Cylindrical shell area: $\frac{2\pi x}{(x+1)(x+2)}$. $V = \int_0^1 \frac{2\pi x}{(x+1)(x+2)}dx =$

$$= 2\pi(2\ln(x+2) - \ln(x+1))\Big|_0^1 = 2\pi(2\ln 3 - 3\ln 2) \approx 0.74005.$$

(b) $y'(x) = 2x - \frac{1}{8x}$; $\sqrt{1 + y'^2} = \frac{1+16x^2}{8x}$. Arc-length $= \int_1^4 \frac{1+16x^2}{8x}dx = (\frac{\ln x}{8} + x^2)\Big|_1^4 = 15 + \frac{\ln 2}{4} \approx 15.173.$

(c) $\rho = x\,e^{-x}$; $x\rho = x^2 e^{-x}$. Center of mass: $\bar{x} = M/m = \frac{2 - 5e^{-1}}{1 - 2e^{-1}} \approx .6078$. Mass: $m = \int_0^1 \rho\,dx = -(x+1)e^{-x}\Big|_0^1 = 1 - 2e^{-1} \approx .2642$. Moment: $M = \int_0^1 x\rho\,dx = -(x^2+2x+2)e^{-x}\Big|_0^1 = 2 - 5e^{-1} \approx .1606$.

3.(a) **(i)** $y = C_1 e^{3x} + C_2 e^x$. **(ii)** $y = e^{2x}(C_1 + C_2 x)$.

(b) Particular solution: $y_p = -2\sin x + \cos x$.

General solution: $y = -2\sin x + \cos x + C_1 e^{3x} + C_2 e^x$. **(c)** $y = -2\sin x + \cos x - 2e^{3x} + 6e^x$.

4.(a) Answer: $f(-1+h, 2+k) \approx -2 + 4h + k - 4h^2 - 4hk - k^2$.

$f_x = y/(x+y) - xy/(x+y)^2$, $f_{xx} = -2y/(x+y)^2 + 2xy/(x+y)^3$,

$f_y = x/(x+y) - xy/(x+y)^2$, $f_{yy} = -2x/(x+y)^2 + 2xy/(x+y)^3$, $f_{xy} = 2xy/(x+y)^3$.

(b) $n = 5$, $\sum_{k=1}^5 x_k = 10$, $\sum_{k=1}^5 x_k^2 = 30$, $\sum_{k=1}^5 y_k = 16 + A + B$, $\sum_{k=1}^5 x_k y_k = 6 + A + 2B$.

$$a = \frac{n\cdot(6 + A + 2B) - 10\cdot(16 + A + B)}{n\cdot 30 - 10^2} = \frac{-(26 + A)}{10}, \quad b = \frac{(16 + A + B) - a\cdot 10}{n} = \frac{42 + 2A + B}{5}.$$

Combining this with $a = -3$ and $b = 12$ yields the Answer: $\boxed{A = 4,\ B = 10}$.

5.(a) **(i)** $x = [-1, 6, 4]^T$. **(ii)** From $\left[\begin{array}{ccc|c} 1 & 0 & -3/2 & -7 \\ 0 & 1 & -2 & -2 \end{array}\right]$ obtain $x = [-7 + (3/2)t,\ -2 + 2t,\ t]^T$.

(iii) From $\left[\begin{array}{ccc|c} 1 & 1/2 & -5/2 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right]$ obtain $x = [-8 - (1/2)t_1 + 5/2t_2,\ t_1,\ t_2]^T$.

(b) $A^{-1} = \begin{bmatrix} 2/9 & -5/9 & -2/9 \\ 1/6 & -1/6 & -1/6 \\ 0 & 2/7 & 1/7 \end{bmatrix}$.