MA4002 Final Exam Answers, Spring 2009

1.(a) Velocity: $v(t) = 20 + \int_0^t \sin(0.2s)\,ds = 25 - 5\cos(0.2t)$. Distance $s(T) = \int_0^T v(t)dt = 25T - 25\sin(0.2T)$ m and $s(30) = \boxed{750 - 25\sin(6) \approx 756.9853875\,\text{m}}$.

(b) The cross-sectional area: $\frac{\pi}{(x+3)^2}$.

$$V = \pi \int_0^4 \frac{dx}{(x+3)^2} = \pi\Big(-\frac{1}{x+3}\Big)\Big|_0^4 = \frac{4\pi}{21} \approx .5983986.$$

(c) Integrating by parts yields $I_n = 2^{-n/2} + (n-1)\int_0^{\pi/4} \sin^{n-2} x \cos^2 x\,dx$. Invoking $\cos^2 x = 1 - \sin^2 x$, we get the desired reduction formula. Next, $I_0 = \dfrac{\pi}{4}$ implies $I_2 = \frac14 + \frac{\pi}{8}$ and $I_6 = \frac14 + \frac{3\pi}{32}$.

(d) $f_x = [y + xy + 1]e^x$, $f_y = x\,e^x$, $f_{xx} = [2y + xy + 1]e^x$, $f_{yy} = 0$, $f_{xy} = [1 + x]e^x$.

(e) $x_n = 0.2n$. Start with $y_0 = 0$. $y_{n+1} = y_n + 0.1[e^{x_n \cdot y_n} + e^{x_{n+1}\cdot y^\star_{n+1}}]$, where $y^\star_{n+1} = y_n + 0.2e^{x_n\cdot y_n}$. Now $y_1^\star = 0 + .2\cdot 1 = .2$; $y(0.2) \approx y_1 = 0 + .1[1 + e^{0.2\cdot 0.2}] \approx .2040810774$. $y_2^\star = .4124132068$, $y(0.4) \approx y_2 \approx .4261823595$. $y_3^\star \approx .6633554727$, $y(0.6) \approx y_3 \approx .6936552968$.

(f) Integrating factor: $v = \exp\{\int \frac{x}{x^2+1}\,dx\} = \sqrt{x^2+1}$. Then $(\sqrt{x^2+1}\cdot y)' = \frac{1}{2\sqrt{x^2+1}}$ and therefore $\sqrt{x^2+1}\cdot y = \frac12 \ln(x + \sqrt{x^2+1}) + C$. By $y(0) = 2$ we get $C = 2$ and $\boxed{y = \frac{1}{2\sqrt{x^2+1}}[\ln(x + \sqrt{x^2+1}) + 4]}$.

(g) 18.

(h) By the Extreme-Value Theorem, $\exists\, A, B \in [a,b]\ :\ f(A) = \min\limits_{[a,b]} f$ and $f(B) = \max\limits_{[a,b]} f$. Furthermore, we have $f(A) \le \bar f \le f(B)$, where $\bar f = (b-a)^{-1}\int_a^b f(x)\,dx$. Finally, by the Intermediate-Value Theorem, $\exists\, c$ between A and B such that $f(c) = \bar f$.

2.(a) Cylindrical shell area: $2\pi x[1 - x^2]$. $V = \int_0^1 2\pi x[1-x^2]dx = 2\pi(\frac{x^2}{2} - \frac{x^4}{4})\Big|_0^1 = \frac{\pi}{2} \approx 1.57$.

(b) $y'(x) = \dfrac{x^3}{8} - \dfrac{2}{x^3}$; $\sqrt{1 + y'^2} = \dfrac{2}{x^3} + \dfrac{x^3}{8}$. Arc-length $= \displaystyle\int_1^4 (\frac{2}{x^3} + \frac{x^3}{8})dx = (-\frac{1}{x^2} + \frac{x^4}{32})\Big|_1^4 = \frac{285}{32} = 8.90625.$

(c) $\rho = \dfrac{1}{(x+1)(4-x)}$; $x\rho = \dfrac{x}{(x+1)(4-x)}$. Center of mass: $\bar x = M/m = \dfrac{4\ln 2 - \ln 3}{\ln 3 + \ln 2} \approx .934264$.

Mass: $m = \displaystyle\int_0^1 \rho\,dx = \frac15[\ln(x+1) - \ln(4-x)]\Big|_0^2 = \frac15[\ln 3 + \ln 2] \approx .35835$. Moment: $M = \displaystyle\int_0^2 x\rho\,dx = \frac15[-4\ln(4-x) - \ln(x+1)]\Big|_0^2 = \frac15[4\ln 2 - \ln 3] \approx .334795.$

3.(a) **(i)** $y = C_1\cos(2x) + C_2\sin(2x)$. **(ii)** $y = e^{-x}(C_1 + C_2 x)$.

(b) Look for a particular solution in the form $y_p = x(a\cos(2x) + b\sin(2x))$, which yields $y_p = -x\cos(2x)$. General solution: $y = -x\cos(2x) + C_1\cos(2x) + C_2\sin(2x)$.

(c) $y = -x\cos(2x) + 2\cos(2x) + \sin(2x)$.

4.(a) Answer: $f(1+h, 2+k) \approx \frac13 h - \frac29 h^2 - \frac19 hk$.

$f_x = \frac{1}{x^2+y} - \frac{2(x-1)x}{(x^2+y)^2}$, $f_{xx} = \frac{-4x - 2(x-1)}{(x^2+y)^2} + \frac{8(x-1)x^2}{(x^2+y)^3}$,

$f_y = \frac{-(x-1)}{(x^2+y)^2}$, $f_{yy} = \frac{2(x-1)}{(x^2+y)^3}$, $f_{xy} = \frac{3x^2 - y - 4x}{(x^2+y)^3}$.

(b) $n = 6$, $\displaystyle\sum_{k=1}^6 x_k = 3$, $\displaystyle\sum_{k=1}^6 x_k^2 = 19$, $\displaystyle\sum_{k=1}^6 y_k = 5 + A + B$, $\displaystyle\sum_{k=1}^6 x_k y_k = 1 + 3B$.

$$a = \frac{n\cdot(1+3B) - 3\cdot(5+A+B)}{n\cdot 19 - 3^2} = \frac{-3 + 5B - A}{35}, \qquad b = \frac{(5+A+B) - a\cdot 3}{n} = \frac{92 + 19A + 10B}{105}.$$

Combining this with $a = 1$ and $b = 2$ yields the Answer: $\boxed{A = 2,\ B = 8}$.

5.(a) **(i)** $x = [2,\,7,\,4]^T$. **(ii)** From $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -6 & -7 \end{array}\right]$ obtain $x = [t,\,-7+6t,\,t]^T$.

(iii) From $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 2 \end{array}\right]$ obtain $x = [1 - t_1,\, 2 + t_1 + t_2,\, t_1,\, t_2]^T$. **(b)** $A^{-1} = \begin{bmatrix} 15 & 9/2 & -7 \\ 1 & 1/2 & -1/2 \\ 2 & 1/2 & -1 \end{bmatrix}$.