MA4002 Final Exam Answers, Spring 2011

1.(a) Velocity: $v(t) = 3 + \int_0^t \frac{ds}{(s+1)^2} = 4 - \frac{1}{t+1}$. Distance $s(T) = \int_0^T v(t)dt = 4T - \ln(T+1)\,\mathrm{m}$ and $s(15) = \boxed{60 - \ln 16 \approx 57.22741128\,\mathrm{m}}$.

(b) (i) The cross-sectional area: $\pi(x^2)^2$. $\quad V = \pi\int_0^1 x^4\,dx = \frac{1}{5}\pi x^5|_0^1 = \frac{1}{5}\pi \approx 0.6283185308$.

(ii) Using cylindrical shells: $V = \int_0^1 2\pi x(x^2)\,dx = 2\pi\cdot\frac{1}{4}x^4|_0^1 = \frac{1}{2}\pi \approx 1.570796327$.

(c) Integrating by parts using $u = (\ln x)^n$ and $dv = x\,dx$ yields the reduction formula $I_n = \int_1^e x\,(\ln x)^n\,dx = \frac{x^2}{2}\,(\ln x)^n|_1^e - \frac{n}{2}I_{n-1} = \frac{e^2}{2} - \frac{n}{2}I_{n-1}$. Next, $I_0 = \frac{e^2}{2} - \frac{1}{2} \approx 3.194528048$ implies $I_1 = \frac{e^2}{4} + \frac{1}{4} \approx 2.097264025$ and $I_2 = \frac{e^2}{4} - \frac{1}{4} \approx 1.597264025$.

(d) $f_x = y^2\,e^{x\,y^2}$, $f_y = 2xy\,e^{x\,y^2}$, $f_{xx} = y^4e^{x\,y^2}$, $f_{yy} = [2x + 4x^2y^2]e^{x\,y^2}$, $f_{xy} = [2y + 2xy^3]e^{x\,y^2}$.

(e) $x_n = 0.2n$. Start with $y_0 = 2$. $\quad y_{n+1} = y_n + \frac{1}{2}\,0.2\,[\ln(2y_n - x_n) + \ln(2y^\star_{n+1} - x_{n+1})]$, where $y^\star_{n+1} = y_n + 0.2\,\ln(2y_n - x_n)$. Now $y^\star_1 = 2 + 0.2\cdot\ln 4 \approx 2.277258872$; $\quad y(0.2) \approx y_1 \approx 2.285750823$. $y^\star_2 = 2.580772138$, $\quad y(0.4) \approx y_2 \approx 2.589318685$.

(f) Rewrite as $y' + \frac{3}{x}y = -\frac{2}{x^2}$ so the integrating factor: $v = \exp\{\int \frac{3}{x}\,dx\} = x^3$. Then $(x^3\cdot y)' = -2x$ and therefore $x^3\cdot y = -x^2 + C$ so $y = -\frac{1}{x} + \frac{C}{x^3}$. By $y(1) = 4$ we get $C = 5$ and $\boxed{y = -\frac{1}{x} + \frac{5}{x^3}}$.

(g) 95 and $2\cdot 95 = 190$.

(h) A calculation yields $\begin{vmatrix} a_1 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & 0 & c_3 & 0 \\ d_1 & 0 & d_3 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 & b_4 \\ 0 & c_3 & 0 \\ 0 & d_3 & d_4 \end{vmatrix} = a_1\,b_2 \begin{vmatrix} c_3 & 0 \\ d_3 & d_4 \end{vmatrix} = a_1\,b_2\,c_3\,d_4$.

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2.(a) For bowl of radius r the height is r^2 and using cylindrical shell area $2\pi x[r^2 - x^2]$, one gets $V(r) = \int_0^r 2\pi x[r^2 - x^2]dx = 2\pi(r^2\frac{x^2}{2} - \frac{x^4}{4})\Big|_0^r = \frac{\pi r^4}{2}$ so the full volume $\boxed{V(2) = 8\pi \approx 25.1327}$.
The bowl is half full when $V(r) = \frac{1}{2}V(2)$ or $\frac{\pi r^4}{2} = \frac{1}{2}8\pi$ so the height $\boxed{h = r^2 = \sqrt{8} = 2\sqrt{2} \approx 2.828427}$.

(b) $y'(x) = 3\sqrt{2x+3}$; $\sqrt{1 + y'^2} = \sqrt{18x + 28}$. Arc-length: $= \int_0^1 \sqrt{18x+28}\,dx = \frac{1}{27}(18x+28)^{3/2}|_0^1$
$\boxed{= \frac{1}{27}[46^{3/2} - 28^{3/2}] \approx 6.067596510.}$

(c) $\rho = \dfrac{1}{(x+2)(x+1)} = \dfrac{1}{x+1} - \dfrac{1}{x+2}$; $\quad x\rho = \dfrac{x}{(x+2)(x+1)} = \dfrac{2}{x+2} - \dfrac{1}{x+1}$. Center of mass: $\bar{x} = M/m = \dfrac{2\ln 2 - \ln 3}{\ln 3 - \ln 2} \approx 0.7095112897$. Mass: $m = \displaystyle\int_0^2 \rho\,dx = [\ln(x+1) - \ln(x+2)]|_0^2 = \ln 3 - \ln 2 \approx 0.4054651084$. Moment: $M = \displaystyle\int_0^2 x\rho\,dx = [2\ln(x+2) - \ln(x+1)]|_0^2 = 2\ln 2 - \ln 3 \approx 0.287682072$.

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3.(a) (i) Roots: $4 \pm i$ so $\quad y = [C_1\cos(x) + C_2\sin(x)]\,e^{4x}$. **(ii)** Roots: 3, -1 so $\quad y = C_1e^{3x} + C_2e^{-x}$.

(b) Look for a particular solution in the form $\; y_p = axe^{3x}$, which yields $\; y_p = 2xe^{3x}$.
General solution: $\; y = 2xe^{3x} + C_1e^{3x} + C_2e^{-x}$. **(c)** $\; y = 2x\,e^{3x} + e^{3x} + 3e^{-x}$.

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4.(a) Answer: $f(1+h, k) \approx 1 - h + k + h^2 - 3hk + \frac{5}{2}k^2$.

$f_x = -\frac{e^{-y}}{(x-2y)^2}$, $\qquad f_{xx} = \frac{2e^{-y}}{(x-2y)^3}$,

$f_y = -\frac{e^{-y}}{x-2y} + \frac{2e^{-y}}{(x-2y)^2}$, $\qquad f_{yy} = \frac{e^{-y}}{x-2y} - \frac{4e^{-y}}{(x-2y)^2} + \frac{8e^{-y}}{(x-2y)^3}$, $\qquad f_{xy} = \frac{e^{-y}(x-2y-4)}{(x-2y)^3}$.

(b) $n = 6$, $\displaystyle\sum_{k=1}^6 x_k = 8$, $\displaystyle\sum_{k=1}^6 x_k^2 = 24$, $\displaystyle\sum_{k=1}^6 y_k = 7 + A + B$, $\displaystyle\sum_{k=1}^6 x_ky_k = -1 + A + 3B$.

$$a = \frac{n\cdot(-1 + A + 3B) - 8\cdot(7 + A + B)}{n\cdot 24 - 8^2} = \frac{-31 - A + 5B}{40}, \qquad b = \frac{(7 + A + B) - a\cdot 8}{n}.$$

Combining this with $a = -1$ and $b = 3$ yields the Answer: $\boxed{A = 4,\ B = -1}$.

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5.(a) **(i)** $x = [7, -1, -2]^T$. **(ii)** $x = [7, -1, -2]^T$. **(iii)** From $\left[\begin{array}{cccc|c} 1 & 0 & -2 & -3 & -2 \\ 0 & 1 & -1 & -1 & -1 \end{array}\right]$ obtain $x = [-2 + 2t_1 + 3t_2,\ -1 + t_1 + t_2,\ t_1,\ t_2]^T$.

(b) $A^{-1} = \begin{bmatrix} -6 & -\frac{13}{3} & \frac{2}{3} & 0 \\ -\frac{31}{2} & -11 & \frac{3}{2} & 0 \\ -10 & -7 & 1 & 0 \\ -20 & -14 & 2 & 1 \end{bmatrix}$.

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