MA4002 Final Exam Answers, Spring 2013

1.(a) Velocity: $v(t) = 15 + \int_0^t (10 - 3\sqrt{s})ds = 15 + 10t - 2t^{3/2}$.
Distance $s(T) = \int_0^T v(t)dt = 15T + 5T^2 - \frac{4}{5}T^{5/2}$m and $s(5) = 200 - 20\sqrt{5} \approx \boxed{155.2786\,\text{m}}$.
(b) Intercepts: $x = 0, 1$. **(i)** The cross-sectional area: $\pi[x]^2 - \pi[x^2]^2$. $V = \pi \int_0^1 (x^2 - x^4)\,dx = \frac{2}{15}\pi \approx 0.418879$. **(ii)** Using cylindrical shells: $V = \int_0^1 2\pi x\,[x - x^2]\,dx = \frac{\pi}{6} \approx 0.523598$.
(c) Integrating by parts using $u = (\ln x)^n$ and $dv = dx$ yields the reduction formula
$I_n = \int_1^e (\ln x)^n\,dx = x\,(\ln x)^n|_1^e - n \cdot I_{n-1} = \boxed{e - n \cdot I_{n-1}}$. Next, $I_0 = e - 1 \approx 1.71828$ implies
$I_1 = e - 1\,I_0 = 1$, $I_2 = e - 2\,I_1 = e - 2 \approx 0.71828$, and $I_3 = e - 3\,I_2 = 6 - 2e \approx 0.563436$.
(d) $f_x = 3x^2\,\cos(x^3 - y)$, $f_y = -\cos(x^3 - y)$, $f_{xx} = 6x\,\cos(x^3 - y) - 9x^4\,\sin(x^3 - y)$, $f_{yy} = -\sin(x^3 - y)$, $f_{xy} = 3x^2\,\sin(x^3 - y)$.
(e) $x_n = 0.2n$. Start with $y_0 = 2$. $y_{n+1} = y_n + \frac{1}{2}0.2\,[\sqrt{x_n + y_n^2} + \sqrt{x_{n+1} + [y^\star_{n+1}]^2}]$, where $y^\star_{n+1} = y_n + 0.2\sqrt{x_n + y_n^2}$. Now $y_1^\star \approx 2.4$; $y(0.2) \approx y_1 \approx 2.444131112$. $y_2^\star = 2.941072835$, $y(0.4) \approx y_2 \approx 2.993432647$.
(f) Rewrite as $y' + \frac{1}{x+1}y = \frac{2x-1}{x+1}$ so the integrating factor: $v = \exp\{\int \frac{1}{x+1}\,dx\} = x+1$. So $([x+1] \cdot y)' = 2x - 1$ and therefore $[x+1] \cdot y = x^2 - x + C$ so $y = \frac{x^2-x+C}{x+1}$. By $y(0) = 3$ we get $C = 3$ and $\boxed{y = \frac{x^2-x+3}{x+1}}$.
(g) -3 and $6 \cdot (-3) = -18$.

(h) An integration by parts using $u = f(x)$ and $dv = dx$ with $v = x - x_1$ yields:
$\int_{x_0}^{x_1} f(x)\,dx = f(x) \cdot (x - x_1)|_{x_0}^{x_1} - \int_{x_0}^{x_1} (x - x_1)f'(x)\,dx = 0 - f(x_0) \cdot (-h) - \int_{x_0}^{x_1} (x - x_1)f'(x)\,dx$. The desired relation follows.

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2.(a) The glass height is $1 - \cos(\frac{\pi}{3}) = \frac{1}{2}$ and using cylindrical shell area $2\pi x[(\frac{1}{2}) - (1 - \cos(\frac{\pi x}{3}))] = 2\pi x[-\frac{1}{2} + \cos(\frac{\pi x}{3})]$, one gets $V = \int_0^1 2\pi x[-\frac{1}{2} + \cos(\frac{\pi x}{3})]\,dx = \boxed{\frac{-\pi^2 - 18 + 6\sqrt{3}\pi}{2\pi} \approx 0.760567}$.
(b) $x'(t) = 1 - \cos t$, $y'(t) = \sin t$; $\sqrt{x'^2 + y'^2} = \sqrt{2 - 2\cos t} = 2|\sin\frac{t}{2}|$.
Arc-length: $= \int_0^\pi 2\sin\frac{t}{2}\,dt = -4\cos\frac{t}{2}|_0^\pi = 4$.
(c) $\rho = \frac{1}{x^2+4}$; $x\rho = \frac{x}{x^2+4} = \frac{1}{2}\frac{(x^2+4)'}{x^2+4}$. Center of mass: $\bar{x} = M/m = \frac{\ln 5 + \ln 2}{\tan^{-1} 3} \approx 1.843475$. Mass: $m = \int_0^6 \rho\,dx = \frac{1}{2}\tan^{-1}(\frac{x}{2})|_0^6 = \frac{1}{2}\tan^{-1} 3 \approx 0.62452$. Moment: $M = \int_0^6 x\rho\,dx = \frac{1}{2}\ln(x^2 + 4)|_0^6 = \frac{1}{2}[\ln 40 - \ln 4] \approx 1.15129$.

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3.(a) **(i)** Roots: $1, 2$ so $y = C_1\,e^x + C_2\,e^{2x}$. **(ii)** Roots: $\frac{1}{2}, \frac{1}{2}$ so $y = [C_1\,x + C_2]\,e^{x/2}$.
(b) Look for a particular solution $y_p = a + bx^2\,e^{x/2}$, which yields $y_p = 5 - \frac{1}{4}x^2\,e^{x/2}$.
General solution: $y = 5 - \frac{1}{4}x^2\,e^{x/2} + [C_1\,x + C_2]\,e^{x/2}$. **(c)** $y = 5 - \frac{1}{4}x^2\,e^{x/2} + [-x - 4]\,e^{x/2}$.

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4.(a) Answer: $f(5+h, 1+k) \approx 32 + 20\,h - 40\,k + \frac{15}{4}\,h^2 - 15\,kh - 5\,k^2$.
$f_x = \frac{5}{2}(x - y^2)^{3/2}$, $f_{xx} = \frac{15}{4}(x - y^2)^{1/2}$, $f_y = -5y\,(x - y^2)^{3/2}$, $f_{xy} = -\frac{15}{2}\,y\,(x - y^2)^{1/2}$, $f_{yy} = 15y^2\,(x - y^2)^{1/2} - 5\,(x - y^2)^{3/2}$; $f_x(5,1) = \frac{5}{2}\,2^3 = 20$, $f_{xx}(5,1) = \frac{15}{4}\,2 = \frac{15}{2}$, $f_y(5,1) = -5 \cdot 1 \cdot 2^3 = -40$, $f_{xy}(5,1) = -\frac{15}{2} \cdot 1 \cdot 2 = -15$, $f_{yy}(5,1) = 15 \cdot 1^2 \cdot 2 - 5 \cdot 2^3 = -10$.

(b) $n = 5$, $(\ln x, \ln y) \approx (0, 3.2188758)$, $(0.693147, 2.70805)$, $(1.09861, 1.38629)$, $(1.386294, 0)$, $(1.6094379, 0.693147)$. $\sum_{k=1}^5 \ln x_k \approx 4.78749$, $\sum_{k=1}^5 (\ln x_k)^2 \approx 6.1995$, $\sum_{k=1}^5 \ln y_k \approx 8.006367$, $\sum_{k=1}^5 \ln x_k \cdot \ln y_k \approx 4.51565$. $\alpha \approx \frac{n \cdot (4.51565) - (4.78749) \cdot (8.006367)}{n \cdot (6.1995) - (4.78749)^2} \approx \boxed{-1.95013}$,
$\ln k \approx \frac{(8.006367) - \alpha \cdot (4.78749)}{n} \approx 3.4685$, so $k = e^{\ln k} \approx \boxed{32.0895}$.

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5.(a) (i) $x = [-63, -10, -14]^T$. **(ii)** From $\left[\begin{array}{ccc|c} 1 & 0 & -8 & 49 \\ 0 & 1 & -1 & 4 \end{array}\right]$ obtain $x = [49 + 8t_1,\ 4 + t_1, t_1]^T$.

(b) From $\left[\begin{array}{rrrr|rrrr} 1 & -3 & 0 & -4 & 1 & 0 & 0 & 0 \\ -3 & 8 & 1 & 7 & 0 & 1 & 0 & 0 \\ 2 & -4 & -3 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & -2 & 5 & 0 & 0 & 0 & 1 \end{array}\right]$ get $\left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 80 & 31 & -3 & 20 \\ 0 & 1 & 0 & 0 & 33 & 13 & -1 & 8 \\ 0 & 0 & 1 & 0 & 11 & 4 & -1 & 3 \\ 0 & 0 & 0 & 1 & -5 & -2 & 0 & -1 \end{array}\right]$,

and then $A^{-1} = \left[\begin{array}{rrrr} 80 & 31 & -3 & 20 \\ 33 & 13 & -1 & 8 \\ 11 & 4 & -1 & 3 \\ -5 & -2 & 0 & -1 \end{array}\right]$.

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