MA4002 Final Exam Answers, Spring 2017

1.(a) Velocity: $v(t) = 7 + \int_0^t \frac{1}{\sqrt{s+9}} ds = 1 + 2\sqrt{t+9}$.
Distance $s(T) = \int_0^T v(t)dt = T + \frac{4}{3}[(T+9)^{3/2} - 3^3]$m and $s(4) = \frac{52\sqrt{13}}{3} - 32 \approx \boxed{30.49622209\,\text{m}}$.
(b) Intercepts: $x = 1, 3$. Using cylindrical shells:
$V = \int_1^3 2\pi x\,[(6x - x^2 - 3) - 2x]\,dx = \pi(\frac{8}{3}x^3 - \frac{1}{2}x^4 - 3x^2)|_1^3 = \frac{16\pi}{3} \approx 16.75516082$.
(c) Integrating by parts using $u = \ln^n x$ and $dv = dx$ yields the reduction formula
$I_n = \int_1^{e^2} \ln^n x\,dx = x\ln^n x|_1^{e^2} - n \cdot I_{n-1} = \boxed{2^n e^2 - n \cdot I_{n-1}}$. Next, $I_0 = e^2 - 1 \approx 6.389056099$ implies $I_1 = 2^1 e^2 - 1 \cdot [e^2 - 1] = e^2 + 1 \approx 8.389056099$, and $I_2 = 2^2 e^2 - 2 \cdot [e^2 + 1] = 2e^2 - 2 \approx 12.77811220$.
(d) $f = \cos(xy - x)$, $f_x = -(y-1)\sin(xy - x)$, $f_y = -x\sin(xy - x)$,
$f_{xx} = -(y-1)^2\cos(xy - x)$, $f_{yy} = -x^2\cos(xy - x)$, $f_{xy} = -x(y-1)\cos(xy - x) - \sin(xy - x)$.
(e) $x_n = 0.3n$. Start with $y_0 = 2$. $y_{n+1} = y_n + \frac{1}{2}0.3\,[\exp(x_n y_n - y_n) + \exp(x_{n+1}y^*_{n+1} - y^*_{n+1})]$, where $y^\star_{n+1} = y_n + 0.3\,\exp(x_n y_n - y_n)$. Now $y_1^\star \approx 2.040600585$; $y(0.3) \approx y_1 \approx 2.056253377$. $y_2^\star = 2.127375974$, $y(0.6) \approx y_2 \approx 2.155866013$, $y_3^\star \approx 2.282517114$, $y(0.9) \approx y_3 \approx 2.338580147$.
(f) Rewrite as $y' - \frac{5}{x}y = \frac{15}{x}$ so the integrating factor: $v = \exp\{-\int \frac{5}{x}\,dx\} = x^{-5}$. So $(x^{-5} \cdot y)' = 15x^{-6}$ and therefore $x^{-5} \cdot y = -3x^{-5} + C$ so $\boxed{y = -3 + Cx^5}$. The initial condition yields $C = 10$ and $\boxed{y = -3 + 10x^5}$. **(g)** 42 and $-2 \cdot 42 = -84$.
(h) For $x > 0$ we have $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$, while for $x < 0$ we have $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-x)' = \frac{1}{x}$. Therefore $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all $x \neq 0$. The desired result follows.

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2.(a) Cylindrical shell area: $2\pi x[\sin\frac{\pi x}{2} - x]$.
$V = 2\pi\int_0^1 x[\sin\frac{\pi x}{2} - x]\,dx = 2\pi[-x\frac{2}{\pi}\cos\frac{\pi x}{2} + \frac{4}{\pi^2}\sin\frac{\pi x}{2} - \frac{1}{3}x^3]\Big|_0^1 = 0 + \frac{8}{\pi} - \frac{2}{3}\pi \approx 0.4520839864$.
(b) $y'(x) = -\dfrac{x}{\sqrt{4-x^2}}$. $\sqrt{1+y'^2} = \dfrac{2}{\sqrt{4-x^2}}$.
Arc-length: $s = \displaystyle\int_0^1 \frac{2}{\sqrt{4-x^2}}\,dx = 2\,\sin^{-1}(\tfrac{x}{2})\Big|_0^1 = \tfrac{1}{3}\pi \approx 1.047197551$.
(c) $\rho = \frac{1}{x^2+4x+3} = \frac{1}{(x+1)(x+3)} = \frac{1}{2}\Big(\frac{1}{x+1} - \frac{1}{x+3}\Big)$; $x\rho = \frac{x}{(x+1)(x+3)} = \frac{1}{2}\Big(\frac{3}{x+3} - \frac{1}{x+1}\Big)$.
Mass: $m = \int_0^2 \rho\,dx = \frac{1}{2}(\ln(x+1) - \ln(x+3))|_0^2 = \ln 3 - \frac{1}{2}\ln 5 \approx 0.2938933330$.
Moment: $M = \int_0^2 x\rho\,dx = \frac{1}{2}(3\ln(x+3) - \ln(x+1))|_0^2 = \frac{3}{2}\ln 5 - 2\ln 3 \approx 0.216932290$.
Center of mass: $\boxed{\bar{x} = M/m = \frac{\frac{3}{2}\ln 5 - 2\ln 3}{\ln 3 - \frac{1}{2}\ln 5} \approx 0.7381327361}$.

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3.(a) **(i)** Roots: $\pm 3i$ so $y = C_1\cos(3x) + C_2\sin(3x)$.
(ii) Roots: 1, 4 so $y = C_1 e^x + C_2 e^{4x}$.
(b) **(i)** Look for a particular solution $y_p = A\cos x + B\sin x + C\,e^x$, which yields $y_p = \frac{17}{4}\sin x + \frac{3}{10}e^x$.
General solution: $y = \frac{17}{4}\sin x + \frac{3}{10}e^x + C_1\cos(3x) + C_2\sin(3x)$.
(ii) Look for a particular solution $y_p = A\cos x + B\sin x + C\,x\,e^x$, which yields
$(3A - 5B)\cos x + (3B + 5A)\sin x - 3Ce^x$ so $y_p = 5\cos x + 3\sin x - x\,e^x$.
General solution: $y = 5\cos x + 3\sin x - x\,e^x + C_1 e^x + C_2 e^{4x}$.

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4.(a) Answer: $f(5+h,2+k) \approx 27 + \frac{9}{2}h + 18k + \frac{1}{8}h^2 + kh + \frac{13}{2}k^2 + \cdots$.
$f_x = \frac{3}{2}\sqrt{x+y^2}$, $f_y = 3y\sqrt{x+y^2}$, $f_{xx} = \frac{3}{4}(x+y^2)^{-1/2}$, $f_{xy} = \frac{3}{2}y(x+y^2)^{-1/2}$,
$f_{yy} = 3\sqrt{x+y^2} + 3y^2(x+y^2)^{-1/2}$.
Using $\sqrt{5+2^2} = 3$, one gets $f(5,2) = 27$,
$f_x(5,2) = \frac{9}{2}$, $f_y(5,2) = 18$, $f_{xx}(5,2) = \frac{1}{4}$, $f_{xy}(5,2) = 1$, $f_{yy}(5,2) = 13$.

(b) $n = 5$, $(\ln x, \ln y) \approx (0., 0.6931471806), (1.098612289, 2.302585093), (1.609437912, 2.833213344), (1.945910149, 3.178053830), (2.197224578, 3.583518938)$. $\sum_{k=1}^{5} \ln x_k \approx 6.851184928$, $\sum_{k=1}^{5} (\ln x_k)^2 \approx 12.41160151$, $\sum_{k=1}^{5} \ln y_k \approx 12.59051839$, $\sum_{k=1}^{5} \ln x_k \cdot \ln y_k \approx 21.14753234$.

$$\beta \approx \frac{n \cdot (21.14753234) - (6.851184928) \cdot (12.59051839)}{n \cdot (12.41160151) - (6.851184928)^2} \approx \boxed{1.288269108},$$

$$\ln k \approx \frac{(12.59051839) - \beta \cdot (6.851184928)}{n} \approx 0.7528696988, \qquad \text{so } k = e^{\ln k} \approx \boxed{2.123083894}.$$

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5.(a) **(i)** This system can be reduced to $\left[\begin{array}{ccc|c} 2 & 0 & 3 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so $x = [\frac{5}{2} - \frac{3}{2}t,\ -3,\ t]^T$.

(ii) From the last row of the RRE form of this system $\left[\begin{array}{ccc|c} 2 & 0 & 3 & 5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9 \end{array}\right]$ one concludes that there are $\boxed{\text{NO solutions}}$

(b) From $\left[\begin{array}{cccc|cccc} -1 & 1 & 4 & -3 & 1 & 0 & 0 & 0 \\ 4 & -3 & 5 & 8 & 0 & 1 & 0 & 0 \\ 2 & -2 & -7 & 6 & 0 & 0 & 1 & 0 \\ 8 & -2 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$ get $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -201 & 7 & -110 & -1 \\ 0 & 1 & 0 & 0 & -718 & 25 & -393 & -4 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -170 & 6 & -93 & -1 \end{array}\right]$,
and then $A^{-1} = \begin{bmatrix} -201 & 7 & -110 & -1 \\ -718 & 25 & -393 & -4 \\ 2 & 0 & 1 & 0 \\ -170 & 6 & -93 & -1 \end{bmatrix}$.

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