MA4002 Final Exam Answers, Spring 2018

1.(a) Velocity: $v(t) = 3 + \int_0^t \frac{1}{\sqrt{s+4}} ds = -1 + 2\sqrt{t+4}$.
Distance $s(T) = \int_0^T v(t)dt = -T + \frac{4}{3}[(T+4)^{3/2} - 8]$m and $s(5) = \frac{61}{3} \approx \boxed{20.33333333\,\text{m}}$.
(b) (i) The cross-sectional area: $\pi[1-x^2]^2$.
$V = \pi\int_0^1 [1-x^2]^2\,dx = \pi\big(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\big)\big|_0^1 = \frac{8}{15}\pi \approx 1.675516082$.
(ii) Using cylindrical shells: $V = \int_0^1 2\pi x\,[1-x^2]\,dx = \pi\big(x^2 - \frac{1}{2}x^4\big)\big|_0^1 = \frac{\pi}{2} \approx 1.570796327$.
(c) Integrating by parts using $u = \ln^n x$ and $dv = x^5 dx$ yields the reduction formula
$I_n = \int_1^e x^5 \ln^n x\,dx = \frac{1}{6}x^6\ln^n x\big|_1^e - \frac{n}{6}\cdot I_{n-1} = \boxed{\frac{1}{6}e^6 - \frac{n}{6}\cdot I_{n-1}}$. Next, $I_0 = \frac{1}{6}(e^6-1) \approx 67.07146553$ implies $I_1 = \frac{1}{6}e^6 - \frac{1}{6}\cdot I_0 = \frac{1}{36}(5e^6+1) \approx 56.05955466$,
and $I_2 = \frac{1}{6}e^6 - \frac{2}{6}\cdot I_1 = \frac{1}{108}(13e^6-1) \approx 48.55161404$.
(d) $f = \sqrt{x+y^2}$, $\quad f_x = \frac{1}{2\sqrt{x+y^2}}$, $\quad f_y = \frac{y}{\sqrt{x+y^2}}$,
$f_{xx} = -\frac{1}{4(x+y^2)^{3/2}}$, $\quad f_{yy} = \frac{1}{\sqrt{x+y^2}} - \frac{y^2}{(x+y^2)^{3/2}} = \frac{x}{(x+y^2)^{3/2}}$, $\quad f_{xy} = -\frac{y}{2(x+y^2)^{3/2}}$.
(e) $x_n = 0.2n$. Start with $y_0 = 2$. $\quad y_{n+1} = y_n + \frac{1}{2}\,0.2\,[\ln(x_n+y_n^2) + \ln(x_{n+1} + (y_{n+1}^*)^2)]$, where $y_{n+1}^\star = y_n + 0.2\,\ln(x_n+y_n^2)$. Now $y_1^\star \approx 2.277258872$; $\quad y(0.2) \approx y_1 \approx 2.307008027$. $\quad y_2^\star = 2.648766409$, $y(0.4) \approx y_2 \approx 2.678250709$, $\quad y_3^\star \approx 3.083169276$, $\quad y(0.6) \approx y_3 \approx 3.112022258$.
(f) Rewrite as $y' + \frac{1}{x}y = -\frac{1}{x^2}$ so the integrating factor: $v = \exp\{\int \frac{1}{x}\,dx\} = x$. So $(x\cdot y)' = -\frac{1}{x}$ and therefore $x\cdot y = -\ln x + C$ so $\boxed{y = \frac{1}{x}(C - \ln x)}$. The initial condition yields $C = 8$ and $\boxed{y = \frac{1}{x}(8-\ln x)}$. **(g)** 98 and $2\cdot 98 = 196$.
(h) Example: $AB = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

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2.(a) Cylindrical shell area: $2\pi x\big[\frac{1}{(x+1)(x+2)(x+3)}\big]$.
$V = 2\pi\int_0^1 \frac{x}{(x+1)(x+2)(x+3)}\,dx = 2\pi\big[-\frac{1}{2}\ln|x+1| + 2\ln|x+2| - \frac{3}{2}\ln|x+3|\big]\Big|_0^1 = 2\pi\big(\frac{7}{2}\ln 3 - \frac{11}{2}\ln 2\big) \approx$ 0.2062990842. NOTE: here one uses the partial fraction representation $\frac{x}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$, where a calculation shows: $A = -\frac{1}{2}$, $B = 2$, $C = -\frac{3}{2}$.
(b) $y'(x) = \dfrac{e^x - e^{-x}}{2} = \sinh x$. $\qquad \sqrt{1+y'^2} = \cosh x$.
Arc-length: $\quad s = \displaystyle\int_0^2 \cosh x\,dx = \sinh x\Big|_0^2 = \sinh 2 = \frac{e^2-e^{-2}}{2} \approx 3.626860408$.
(c) $\rho = xe^{-x}$; $\;x\rho = x^2e^{-x}$.
Mass (integrate by parts): $m = \int_0^3 \rho\,dx = -(x+1)e^{-x}\big|_0^3 = 1 - 4e^{-3} \approx 0.8008517265$.
Moment (integrate by parts twice): $M = \int_0^3 x\rho\,dx = x^2(-e^{-x})\big|_0^3 - \int_0^3 (2x)(-e^{-x})\rho\,dx$
$= -(x^2+2x+2)e^{-x}\big|_0^3 = 2 - 17e^{-3} \approx 1.153619838$.
Center of mass: $\boxed{\bar{x} = M/m = \frac{2-17e^{-3}}{1-4e^{-3}} \approx 1.440491167}$.

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3.(a) (i) Roots: 3 and 3 so $y = (C_1 + C_2x)e^{3x}$.
(ii) Roots: 2, 4 so $y = C_1\,e^{2x} + C_2\,e^{4x}$.
(b) (i) Look for a particular solution $y_p = A\,e^{2x} + B\,\cos x + C\,\sin x$, which yields
$A\,e^{2x} + (8B - 6C)\cos x + (6B + 8C)\sin x$ so $y_p = e^{2x} - \frac{2}{5}\cos x + \frac{3}{10}\sin x$.
General solution: $y = e^{2x} - \frac{2}{5}\cos x + \frac{3}{10}\sin x + (C_1 + C_2x)e^{3x}$.
(ii) Look for a particular solution $y_p = A\,x\,e^{2x} + B\,\cos x + C\,\sin x$, which yields
$-2A\,e^{2x} + (7B - 6C)\cos x + (6B + 7C)\sin x$ so $y_p = -\frac{1}{2}xe^{2x} - \frac{7}{17}\cos x + \frac{6}{17}\sin x$.
General solution: $y = -\frac{1}{2}xe^{2x} - \frac{7}{17}\cos x + \frac{6}{17}\sin x + C_1\,e^{2x} + C_2\,e^{4x}$.

4.(a) Answer: $f(1+h,k) \approx 2h + k - h^2 - hk - \frac{1}{2}k^2 + \cdots$.

$f_x = \frac{2x+y}{x^2+xy}$, $\quad f_y = \frac{1}{x+y}$, $\quad f_{xx} = \frac{2(x^2+xy)-(2x+y)^2}{(x^2+xy)^2} = \frac{-2x^2-2xy-y^2}{(x^2+xy)^2}$, $\quad f_{xy} = -\frac{1}{(x+y)^2}$, $\quad f_{yy} = -\frac{1}{(x+y)^2}$.

Using $1^2 + 1 \cdot 0 = 1$ and $1 + 0 = 1$, one gets $f(1,0) = 0$,

$f_x(1,0) = 2, \quad f_y(1,0) = 1, \quad f_{xx}(1,0) = -2, \quad f_{xy}(1,0) = -1, \quad f_{yy}(1,0) = -1.$

(b) $n = 5$, $(\ln x, \ln y) \approx (0.6931471806, 0), (1.386294361, 1.098612289), (1.791759469, 1.791759469), (2.079441542, 1.945910149), (2.302585093, 2.079441542)$. $\displaystyle\sum_{k=1}^{5} \ln x_k \approx 8.253227646$, $\displaystyle\sum_{k=1}^{5} (\ln x_k)^2 \approx 15.23864230$, $\displaystyle\sum_{k=1}^{5} \ln y_k \approx 6.915723449$, $\displaystyle\sum_{k=1}^{5} \ln x_k \cdot \ln y_k \approx 13.56789951$.

$$\beta \approx \frac{n \cdot (13.56789951) - (8.253227646) \cdot (6.915723449)}{n \cdot (15.23864230) - (8.253227646)^2} \approx \boxed{1.332408663},$$

$$\ln k \approx \frac{(6.915723449) - \beta \cdot (8.253227646)}{n} \approx -0.8161897122, \qquad \text{so } k = e^{\ln k} \approx \boxed{0.4421130271}.$$

5.(a) **(i)** This system can be reduced to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so $\boxed{x = [-7, \ -4, \ -1]^T}$.

(ii) This system can be reduced to $\left[\begin{array}{ccc|c} 1 & 0 & 13 & -20 \\ 0 & 1 & 5 & -9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so $\boxed{x = [-20 - 13t, \ -9 - 5t, \ t]^T}$.

(b) From $\left[\begin{array}{cccc|cccc} 1 & -3 & 4 & -2 & 1 & 0 & 0 & 0 \\ 3 & -6 & 10 & -3 & 0 & 1 & 0 & 0 \\ -1 & 3 & -2 & 5 & 0 & 0 & 1 & 0 \\ 2 & 0 & 8 & 10 & 0 & 0 & 0 & 1 \end{array}\right]$ get $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -1 & -3 & 1 \\ 0 & 1 & 0 & 0 & -\frac{8}{3} & \frac{7}{3} & \frac{7}{3} & -1 \\ 0 & 0 & 1 & 0 & -1 & \frac{3}{2} & 2 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & \frac{1}{2} \end{array}\right]$,

and then $A^{-1} = \begin{bmatrix} -1 & -1 & -3 & 1 \\ -\frac{8}{3} & \frac{7}{3} & \frac{7}{3} & -1 \\ -1 & \frac{3}{2} & 2 & -\frac{3}{4} \\ 1 & -1 & -1 & \frac{1}{2} \end{bmatrix}$.
