MA4002 Final Exam Answers, Spring 2019

1.(a) Velocity: $v(t) = 1 + \int_0^t 2(s+1)^{-1/3}ds = -2 + 3(t+1)^{2/3}$.

Distance $s(T) = \int_0^T v(t)dt = -2T + \frac{9}{5}[(T+1)^{5/3} - 1]$m and $s(7) = \frac{209}{5} = \boxed{41.8\,\mathrm{m}}$.

(b) (i) The cross-sectional area: $\pi[1^2 - (x^3)^2]$.

$V = \pi\int_0^1 [1 - x^6]\,dx = \pi\big(x - \frac{1}{7}x^7\big)\big|_0^1 = \frac{6}{7}\pi \approx 2.692793703$.

(ii) Using cylindrical shells: $V = \int_0^1 2\pi x\,[1 - x^3]\,dx = \pi\big(x^2 - \frac{2}{5}x^5\big)\big|_0^1 = \frac{3}{5}\pi \approx 1.884955592$.

(c) Integrating by parts using $u = x^n$ and $dv = e^{x/3}dx$ yields the reduction formula

$I_n = x^n\,3e^{x/3}\big|_0^3 - \int_0^3 (3e^{x/3})(nx^{n-1})dx = \boxed{3^{n+1}e - 3nI_{n-1}}$.

Next, $I_0 = 3(e-1) \approx 5.154845484$ implies $I_1 = 3^2e - 3\cdot 1\cdot I_0 = 9$,

and $I_2 = 3^3e - 3\cdot 2\cdot I_1 = 27e - 54 \approx 19.39360936$.

(d) $f = e^{xy-2}, \quad f_x = y\,e^{xy-2}, \quad f_y = x\,e^{xy-2}$,

$f_{xx} = y^2e^{xy-2}, \quad f_{yy} = x^2e^{xy-2}, \quad f_{xy} = [1+xy]e^{xy-2}$.

(e) Answer: $f(2+h, 1+k) \approx 1 + h + 2k$.

Using the results from part (d), $\quad f(2,1) = 1, \quad f_x(2,1) = 1, \quad f_y(2,1) = 2$.

(f) To solve $y' + \frac{3}{x}y = \frac{2}{x^2}$ note the integrating factor: $v = \exp\{\int \frac{3}{x}\,dx\} = x^3$. So $(x^3\cdot y)' = 2x$ and therefore $x^3\cdot y = x^2 + C$ so $\boxed{y = x^{-1} + Cx^{-3}}$. The initial condition yields $C = 4$ and $\boxed{y = x^{-1} + 4x^{-3}}$.

(g) -34 and $(-2)\cdot(-34) = 68$.

(h) Example: $AB = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

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2.(a) Cylindrical shell area: $2\pi x\big[\frac{x+1}{(x+2)(x+3)(x+4)}\big]$.

$V = 2\pi\int_0^2 \frac{x^2+x}{(x+2)(x+3)(x+4)}\,dx = 2\pi\int_0^2\big(\frac{1}{x+2} - \frac{6}{x+3} + \frac{6}{x+4}\big)\,dx = 2\pi\big(\ln|x+2| - 6\ln|x+3| + 6\ln|x+4|\big)\Big|_0^2 = 2\pi\big(-5\ln 2 + 12\ln 3 - 6\ln 5\big) \approx .3831743698$. NOTE: here one uses the partial fraction representation $\frac{x^2+x}{(x+2)(x+3)(x+4)} = \frac{A}{x+2} - \frac{B}{x+3} + \frac{C}{x+4}$, where a calculation shows: $A = 1$, $B = -6$, $C = 6$.

(b) $y'(x) = 3(2x+2)^{1/2}$. $\qquad \sqrt{1 + y'^2} = \sqrt{18x + 19}$.

Arc-length: $\quad s = \displaystyle\int_0^4 \sqrt{18x+19}\,dx = \frac{2}{3}\cdot\frac{1}{18}(18x+19)^{3/2}\Big|_0^4 = \frac{1}{27}[(18\cdot 4+19)^{3/2} - 19^{3/2}] \approx 29.08391086$.

(c) $\rho = x\sin x$; $\ x\rho = x^2\sin x$.

Mass (integrate by parts): $m = \int_0^\pi \rho\,dx = [\sin x - x\cos x]\big|_0^\pi = \pi \approx 3.141592654$.

Moment (integrate by parts twice): $M = \int_0^\pi x\rho\,dx = x^2(-\cos x)\big|_0^\pi - \int_0^\pi (2x)(-\cos x)\,dx$

$= (-x^2\cos x + 2x\sin x + 2\cos x)\big|_0^\pi = \pi^2 - 4 \approx 5.869604404$.

Center of mass: $\boxed{\bar{x} = M/m = \frac{\pi^2 - 4}{\pi} \approx 1.868353110}$.

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3.(a) (i) Roots: 1 and 3 so $y = C_1e^x + C_2e^{3x}$.

(ii) Roots: $2+i,\ 2-i$ so $y = e^{2x}[C_1\cos x + C_2\sin x]$.

(b) (i) Look for a particular solution $y_p = A\,xe^x + B\cos x + C\sin x$, which yields

$-2A\,e^x + (2B - 4C)\cos x + (4B + 2C)\sin x = 2e^x + 5\sin x$ so $y_p = -xe^x + \cos x + \frac{1}{2}\sin x$.

General solution: $y = -xe^x + \cos x + \frac{1}{2}\sin x + C_1e^x + C_2e^{3x}$.

(ii) Look for a particular solution $y_p = A\,e^x + B\cos x + C\sin x$, which yields

$2A\,e^x + (4B - 4C)\cos x + (4B + 4C)\sin x = 2e^x + 5\sin x$ so $y_p = e^x + \frac{5}{8}\cos x + \frac{5}{8}\sin x$.

General solution: $y = e^x + \frac{5}{8}\cos x + \frac{5}{8}\sin x + e^{2x}[C_1\cos x + C_2\sin x]$.

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4.(a) $f = \frac{1}{x+y^2}$. Answer: $f(h, 1+k) \approx 1 - h - 2k + h^2 + 4hk + 3k^2$.

$f_x = \frac{-1}{(x+y^2)^2}$, $f_y = \frac{-2y}{(x+y^2)^2}$, $f_{xx} = \frac{2}{(x+y^2)^3}$, $f_{xy} = \frac{4y}{(x+y^2)^3}$, $f_{yy} = \frac{-2}{(x+y^2)^2} + \frac{-2y\cdot(-2)2y}{(x+y^2)^3} = \frac{-2x+6y^2}{(x+y^2)^3}$.

Using $0 + 1^2 = 1$, one gets $f(0,1) = 1$,

$f_x(0,1) = -1, \quad f_y(0,1) = -2, \quad f_{xx}(0,1) = 2, \quad f_{xy}(0,1) = 4, \quad f_{yy}(0,1) = 6.$

(b) $n = 5$, $(\ln x,\ \ln y) \approx (0,\ 1.386294361), (0.6931471806,\ 1.791759469), (1.098612289,\ 1.945910149), (1.386294361,\ 2.079441542), (1.609437912,\ 2.140066163)$. $\sum_{k=1}^{5} \ln x_k \approx 4.787491743$, $\sum_{k=1}^{5} (\ln x_k)^2 \approx 6.199504424$, $\sum_{k=1}^{5} \ln y_k \approx 9.343471684$, $\sum_{k=1}^{5} \ln x_k \cdot \ln y_k \approx 9.706775528$.

$\beta \approx \dfrac{n\cdot(9.706775528) - (4.787491743)\cdot(9.343471684)}{n\cdot(6.199504424) - (4.787491743)^2} \approx \boxed{.4707038096}$,

$\ln\alpha \approx \dfrac{(9.343471684) - \beta\cdot(4.787491743)}{n} \approx 1.417996216, \qquad \text{so } \alpha = e^{\ln\alpha} \approx \boxed{4.128838845}$.

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5 NOTE: For detailed evaluations, see the **Maple solutions** attached.

(a) **(i)** This system can be reduced to $\left[\begin{array}{ccc|c} 1 & 0 & 1 & -16 \\ 0 & 1 & -1 & -9 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & -3 & -15 \end{array}\right]$ so, from the third row, $\boxed{\text{NO solutions}}$.

(ii) This system can be reduced to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -21 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -15 \end{array}\right]$ so $\boxed{[x,y,z]^T = [-21,\ -4,\ 5]^T}$.

(b) From $\left[\begin{array}{cccc|cccc} 2 & -1 & 3 & 0 & 1 & 0 & 0 & 0 \\ 4 & -1 & 7 & 1 & 0 & 1 & 0 & 0 \\ -2 & 6 & 3 & 2 & 0 & 0 & 1 & 0 \\ 2 & 1 & 8 & -6 & 0 & 0 & 0 & 1 \end{array}\right]$ get $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{345}{2} & -74 & \frac{35}{2} & -\frac{13}{2} \\ 0 & 1 & 0 & 0 & 107 & -46 & 11 & -4 \\ 0 & 0 & 1 & 0 & -79 & 34 & -8 & 3 \\ 0 & 0 & 0 & 1 & -30 & 13 & -3 & 1 \end{array}\right]$,

and then $A^{-1} = \begin{bmatrix} \frac{345}{2} & -74 & \frac{35}{2} & -\frac{13}{2} \\ 107 & -46 & 11 & -4 \\ -79 & 34 & -8 & 3 \\ -30 & 13 & -3 & 1 \end{bmatrix}$.

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