MA4002 Final Exam Answers, Spring 2022

1.(a) *An object has acceleration* $a(t) = \dfrac{2}{(t+1)^{3/5}}$ *metres/second*^{2} *at time* t*. The initial velocity at time* $t = 0$ *is* $v = 2$ *metres/second. How far does it travel in the first* 6 *seconds?*

Velocity: $v(t) = 2 + \int_0^t 2(s+1)^{-3/5}\,ds = -3 + 5(t+1)^{2/5}$.

Distance $s = \int_0^6 v(t)\,dt = \left(-3s + \frac{25}{7}(s+1)^{7/5}\right)\Big|_0^6$m so $s = -\frac{151}{7} + 25\cdot 7^{2/5} = \boxed{32.876\,\text{m}}$.

(b) *Consider the plane region bounded by the curves* $y = 3x - 2x^2$ *and* $y = x^3$ *for* $x \geq 0$*. Find the volume of each of the* <u>two solids</u> *obtained by rotating this plane region* **(i)** *about the* x*-axis;* **(ii)** *about the* y*-axis.*

To find the intercepts, solve $3x - 2x^2 = x^3$ for $x \geq 0$, which is equivalent to $x(x^2 + 2x - 3) = 0$, with roots 0 and 1, (while the negative root -3 is dropped).

(i) The cross-sectional area: $\pi[(3x - 2x^2)^2 - (x^3)^2]$.

$V = \pi \int_0^1 [9x^2 - 12x^3 + 4x^4 - x^6]\,dx = \pi\left(3x^3 - 3x^4 + \frac{4}{5}x^5 - \frac{1}{7}x^7\right)\Big|_0^1 = \frac{23}{35}\pi \approx 2.064475172.$

(ii) Using cylindrical shells: $V = \int_0^1 2\pi x\,[(3x - 2x^2) - x^3]\,dx = \int_0^1 \pi[6x^2 - 4x^3 - 2x^4]\,dx = \pi\left(2x^3 - x^4 - \frac{2}{5}x^5\right)\Big|_0^1 = \frac{3}{5}\pi \approx 1.884955592.$

(c) *Obtain an iterative reduction formula for* $I_n = \displaystyle\int_0^1 x^n e^{-x/3} dx$*. Evaluate* I_0*. Then, using the reduction formula obtained, evaluate* I_1 *and* I_2*.*

Integrating by parts using $u = x^n$ and $dv = e^{-x/3}dx$ yields the reduction formula

$I_n = x^n\,(-3e^{-x/3})\big|_0^1 - \int_0^1 (-3e^{-x/3})(nx^{n-1})dx = \boxed{-3e^{-1/3} + 3nI_{n-1}}$ for $n \geq 1$.

Next, $I_0 = 3 - 3e^{-1/3} \approx 0.8504060683$

implies $I_1 = -3e^{-1/3} + 3\cdot 1\cdot I_0 = 9 - 12e^{-1/3} \approx 0.401624273$,

and $I_2 = -3e^{-1/3} + 3\cdot 2\cdot I_1 = 54 - 75e^{-1/3} \approx 0.26015170$.

(d) *Find all first and second partial derivatives of* $f(x,y) = \sin(x^2 - y^3)$*.*

$f_x = 2x\cos(x^2 - y^3)$, $\quad f_y = -3y^2\cos(x^2 - y^3)$, $\quad f_{xx} = 2\cos(x^2 - y^3) - 4x^2\sin(x^2 - y^3)$,

$f_{yy} = -6y\cos(x^2 - y^3) - 9y^4\sin(x^2 - y^3)$, $\quad f_{xy} = 6xy^2\sin(x^2 - y^3)$.

(e) *Find the linearization of the function $f(x,y) = \sin(x^2 - y^3)$ about the point (2,1). (You may use the results of part (d).)*

$f(2,1) = \sin(2^2 - 1^3) = \sin 3 \approx 0.1411200081$, $f_x(2,1) = 2 \cdot 2\cos 3 \approx -3.959969986$, $f_y(2,1) = -3 \cdot 1^2 \cos 3 \approx 2.969977490$.

Answer: $f(2+h, 1+k) \approx 0.1411200081 - 3.959969986\, h + 2.969977490\, k$.

(f) *Solve the differential equation* $x\dfrac{dy}{dx} + 5y = \dfrac{4\sin(2x)}{x^3}$ *(for $x > 1$), subject to the initial condition* $y\left(\dfrac{\pi}{4}\right) = 2$.

To solve $y' + \frac{5}{x}y = \frac{4\sin(2x)}{x^4}$, find the integrating factor: $v = \exp\{\int \frac{5}{x}\,dx\} = x^5$.

So $(x^5 \cdot y)' = 4x\sin(2x)$. Therefore (using integration by parts), $x^5 \cdot y = \sin(2x) - 2x\cos(2x) + C$ so $\boxed{y = x^{-5}\sin(2x) - 2x^{-4}\cos(2x) + Cx^{-5}}$. The initial condition yields: $2 = \frac{\sin(\pi/2) - 2(\pi/4)\cos(\pi/2) + C}{(\pi/4)^5}$ so $2(\pi/4)^5 = 1 + C$, so $\boxed{C = \frac{\pi^5}{512} - 1 \approx -0.4023}$ and $\boxed{y = x^{-5}\sin(2x) - 2x^{-4}\cos(2x) + \left(\frac{\pi^5}{512} - 1\right)x^{-5}}$.

(g) *Evaluate the three determinants*

$$\begin{vmatrix} 2 & 3 & 0 \\ 1 & 4 & -2 \\ -1 & 5 & -3 \end{vmatrix}, \quad \begin{vmatrix} 3 & 0 & 1 \\ 4 & -2 & 4 \\ 5 & -3 & 2 \end{vmatrix}, \textit{ and } \begin{vmatrix} 1 & 0 & 0 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 4 & -2 & 4 \\ -1 & 5 & -3 & 2 \end{vmatrix}$$

Answers: $\boxed{11}$ and $\boxed{22}$, and then, using the first row expansion, $1 \cdot 22 - 2 \cdot 11 = \boxed{0}$.

(h) *Prove that* $\displaystyle\int \frac{dx}{x} = \ln|x| + C$ *(for $x \neq 0$) from the definition of the indefinite integral. (Hint: consider the cases of $x > 0$ and $x < 0$.)*

For $x > 0$ we have $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$, while for $x < 0$ we have $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-x)' = \frac{1}{x}$, where we used the chain rule. Therefore $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all $x \neq 0$. The desired result follows.

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2.(a) *A solid of revolution is obtained by rotating about the y-axis the area bounded between* $y = \dfrac{1}{(x+1)(x+2)^2}$ *and the x-axis for $0 \le x \le 3$. Find the volume of the solid obtained.*

Cylindrical shell area: $2\pi x\left[\frac{1}{(x+1)(x+2)^2}\right] = 2\pi\left[\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}\right] = 2\pi\left[\frac{(-1)}{x+1} + \frac{1}{x+2} + \frac{2}{(x+2)^2}\right]$,

where we used the partial fraction representation,

for which a calculation shows: $A = -1$, $B = 1$, $C = 2$.

$V = 2\pi\int_0^3 \frac{x}{(x+1)(x+2)^2}\,dx = 2\pi\int_0^3\left[\frac{(-1)}{x+1} + \frac{1}{x+2} + \frac{2}{(x+2)^2}\right]dx$

$= 2\pi\left(-\ln|x+1| + \ln|x+2| - 2(x+2)^{-1}\right)\Big|_0^3 = 2\pi\left(\ln 5 - 3\ln 2 + \frac{3}{5}\right) \approx 0.8167912820.$

(b) *Find the arc-length of the curve $y = x^{3/2}$ for $0 \le x \le 1$.*

$y'(x) = \dfrac{3}{2}x^{1/2}.$ $\qquad \sqrt{1+y'^2} = \dfrac{1}{2}\sqrt{9x+4}.$

Arc-length: $\quad s = \displaystyle\int_0^1 \frac{1}{2}\sqrt{9x+4}\,dx = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{9}(9x+4)^{3/2}\Big|_0^1 = \frac{1}{27}[(9\cdot 1+4)^{3/2} - 4^{3/2}] =$ $\frac{1}{27}[(13)^{3/2} - 8] \approx 1.439709874.$

(c) *Find the mass and the centre of mass of a rod with mass density $\rho(x) = \ln(x+1)$ for $0 \le x \le 4$.*

$\rho = \ln(x+1); \quad x\rho = x\ln(x+1).$

Mass (using $t = x+1$, integrate by parts with $u = \ln t$):

$m = \int_0^4 \rho\,dx = \int_1^5 \ln t\,dt = [t\ln t - t]\Big|_1^5 = 5\ln 5 - 4 \approx 4.047189560.$

Moment (integrate by parts twice):

$M = \int_0^4 x\rho\,dx = \int_1^5 (t-1)\ln t\,dt = \int_1^5 t\ln t\,dt - m = \left(\frac{1}{2}t^2\ln t - \frac{1}{4}t^2\right)\Big|_1^5 - m = \frac{1}{2}25\ln 5 - \frac{1}{4}\cdot 24 -$ $(5\ln 5 - 4) = \frac{15}{2}\ln 5 - 2 \approx 10.07078434$

Center of mass: $\boxed{\bar{x} = M/m = \frac{\frac{15}{2}\ln 5 - 2}{5\ln 5 - 4} \approx 2.488340166}$.

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3.(a) *Find general solutions of the given differential equations:*

(i) $y'' - 8y' - 9y = 0,$ *(ii)* $y'' - 2y' + 26y = 0$

(i) Roots: 9 and -1 so $y = C_1 e^{9x} + C_2 e^{-x}$.

(ii) Roots: $1 + 5i,\ 1 - 5i$ so $y = e^x[C_1\ \cos(5x) + C_2\ \sin(5x)]$.

(b) *Find a particular solution for each of the given differential equations:*

(i) $y'' - 8y' - 9y = 2e^{-x} - 9x,$ (ii) $y'' - 2y' + 26y = 2e^{-x} - 9x.$

Then find the general solutions of these equations.

(i) Look for a particular solution in the form $y_p = A\,xe^{-x} + (B\,x + C)$, which yields

$-10Ae^{-x} - 9B\,x - (8B + 9C) = 2e^{-x} - 9x$ so $y_p = -\frac{1}{5}\,xe^{-x} + x - \frac{8}{9}$.

General solution: $y = -\frac{1}{5}\,xe^{-x} + x - \frac{8}{9} + C_1 e^{9x} + C_2 e^{-x}$.

(ii) Look for a particular solution $y_p = A\,e^{-x} + (B\,x + C)$, which yields

$29A\,e^{-x} + 26B\,x + (26C - 2B) = 2e^{-x} - 9x$ so $y_p = -\frac{2}{29}e^{-x} - \frac{9}{26}x - \frac{9}{338}$.

General solution: $y = -\frac{2}{29}e^{-x} - \frac{9}{26}x - \frac{9}{338} + e^x[C_1\ \cos(5x) + C_2\ \sin(5x)]$.

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4.(a) *Find the Taylor Series, up to and including quadratic terms, of* $z = f(x,y) = \ln(xy^3 - 1)$ *about the point* $(3,1)$.

Answer: $f(3+h, 1+k) \approx \ln 2 + \frac{1}{2}h + \frac{9}{2}k - \frac{1}{8}h^2 - \frac{3}{4}hk - \frac{45}{8}k^2$.

$f_x = \frac{y^3}{xy^3-1}, \quad f_y = \frac{3xy^2}{xy^3-1}, \quad f_{xx} = \frac{-y^6}{(xy^3-1)^2}, \quad f_{xy} = \frac{3y^2(xy^3-1)-3xy^2(y^3)}{(xy^3-1)^2} = \frac{-3y^2}{(xy^3-1)^2},$

$f_{yy} = \frac{6xy(xy^3-1)-3xy^2(3xy^2)}{(xy^3-1)^2} = \frac{-3x^2y^4-6xy}{(xy^3-1)^2}.$

Using $3 \cdot 1^3 - 1 = 2$, one gets $f(3,1) = \ln 2$,

$f_x(3,1) = \frac{1}{2}, \quad f_y(3,1) = \frac{9}{2}, \quad f_{xx}(3,1) = -\frac{1}{4}, \quad f_{xy}(3,1) = -\frac{3}{4}, \quad f_{yy}(3,1) = -\frac{45}{4}.$

(b) *It is known that the quantities* $z > 0$ *and* $t > 0$ *are related by the formula* $z^\alpha = 2t^\beta$, *with some unknown constants* $\alpha \neq 0$ *and* β. *By writing this as* $\alpha \ln z = \beta \ln t + \ln 2$, *and then as*

$$\ln z = \frac{\beta}{\alpha} \ln t + \frac{\ln 2}{\alpha},$$

one can use the method of *least squares* *to find the best-fit line relating* $\ln z$ *to* $\ln t$ *and hence find an approximation of the constants* α *and* β. *For the given data points*

$(t,\ z) = (1,\ 9),\ (3,\ 2),\ (5,\ 7),\ (7,\ 5),\ (9,\ 3),$

use this method to find an approximation of the constants α *and* β.

$n = 5$, $(\ln t,\ \ln z) \approx (0,\ 2.197224578), (1.098612289,\ 0.6931471806), (1.609437912,\ 1.945910149), (1.945910149,\ 1.609437912), (2.197224578,\ 1.098612289)$. $\sum_{k=1}^{5} \ln t_k \approx 6.851184928$,

$\sum_{k=1}^{5} (\ln t_k)^2 \approx 12.41160151, \quad \sum_{k=1}^{5} \ln z_k \approx 7.544332109, \quad \sum_{k=1}^{5} \ln t_k \cdot \ln z_k \approx 9.439041068.$

$$a \approx \frac{n \cdot (9.439041068) - (6.851184928) \cdot (7.544332109)}{n \cdot (12.41160151) - (6.851184928)^2} \approx \boxed{-0.2971312979},$$

$b \approx \frac{(7.544332109) - a \cdot (6.851184928)}{n} \approx 1.916006716, \qquad$ so $\alpha = \frac{\ln 2}{b} \approx \frac{\ln 2}{1.916006716} \approx \boxed{0.3617665715}$

and $\beta = a \cdot \alpha \approx -0.2971312979 \cdot 0.3617665715 \approx \boxed{-0.1074921709}$.

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5 NOTE: For detailed evaluations, see the **Maple solutions** attached.

(a) *Find all solutions of each system of linear equations:*

$$\text{(i)}\quad \begin{aligned} 2x - y + 2z &= 3 \\ 4x - y + 3z &= 4 \\ -2x - y + z &= 9 \\ 6x + y + 3z &= 9 \end{aligned} \quad ; \quad \text{(ii)}\quad \begin{aligned} 2x - y + 2z &= 6 \\ 4x - y + 3z &= 8 \\ -2x - y &= 2 \\ 6x + y + 2z &= 2 \end{aligned}$$

(i) This system can be reduced to $\left[\begin{array}{ccc|c} 2 & 0 & 0 & -7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so $\boxed{x = -\frac{7}{2},\ y = 6,\ z = 8}$.

(ii) This system can be reduced to $\left[\begin{array}{ccc|c} 2 & 0 & 1 & 2 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so $\boxed{x = 1 - \frac{1}{2}t,\ y = -4 + t,\ z = t}$.

(b) *Find the inverse of a matrix.*

From $\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ -2 & -1 & 2 & -4 & 0 & 1 & 0 & 0 \\ 3 & -1 & -9 & 4 & 0 & 0 & 1 & 0 \\ 4 & 3 & 3 & 17 & 0 & 0 & 0 & 1 \end{array}\right]$ get $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 127 & 117 & 28 & 6 \\ 0 & 1 & 0 & 0 & -80 & -75 & -18 & -4 \\ 0 & 0 & 1 & 0 & 41 & 38 & 9 & 2 \\ 0 & 0 & 0 & 1 & -23 & -21 & -5 & -1 \end{array}\right]$,

and then $A^{-1} = \begin{bmatrix} 127 & 117 & 28 & 6 \\ -80 & -75 & -18 & -4 \\ 41 & 38 & 9 & 2 \\ -23 & -21 & -5 & -1 \end{bmatrix}$.

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