MA4002 Final Exam Answers, Spring 2023

1.(a) *An object has acceleration* $a(t) = \dfrac{1}{(t+1)^2}$ *metres/second*^{2} *at time* t*. The initial velocity at time* $t = 0$ *is* $v = 3$ *metres/second. How far does it travel in the first* 2 *seconds?*

Velocity: $v(t) = 3 + \int_0^t (s+1)^{-2}\,ds = 4 - (t+1)^{-1}$.

Distance $s = \int_0^2 v(t)\,dt = \big(4t - \ln|t+1|\big)\Big|_0^2$m so $\boxed{s = 8 - \ln 3 \approx 6.901387711\,\text{m}}$.

(b) *Consider the plane region bounded by the curves* $y = x + \sin x$ *and* $y = \sin x$ *for* $0 \le x \le 1$*. Find the volume of each of the* <u>two solids</u> *obtained by rotating this plane region* **(i)** *about the* x*-axis;* **(ii)** *about the* y*-axis.*

(i) The cross-sectional area: $\pi[(x+\sin x)^2 - (\sin x)^2] = \pi[x^2 + 2x\sin x]$.

$V = \pi\int_0^1 [x^2 + 2x\sin x]\,dx = \pi\big(\tfrac{1}{3}x^3\big)\Big|_0^1 + 2\pi\Big(x(-\cos x)\Big|_0^1 - \int_0^1(-\cos x)dx\Big)$

$= \tfrac{1}{3}\pi + 2\pi\big(-x\cos x + \sin x\big)\Big|_0^1 = \boxed{\pi\big(\tfrac{1}{3} - 2\cos 1 + 2\sin 1\big) \approx 2.939496169.}$

(ii) Using cylindrical shells:

$V = \int_0^1 2\pi x\,[(x+\sin x) - \sin x]\,dx = \int_0^1 2\pi[x^2]\,dx = \tfrac{2}{3}\pi x^3\Big|_0^1 = \boxed{\tfrac{2}{3}\pi \approx 2.094395103}$.

(c) *Obtain an iterative reduction formula for* $I_n = \displaystyle\int x^n e^{2x} dx$*. Evaluate* I_0*. Then, using the reduction formula obtained, evaluate* I_1 *and* I_2*.*

Integrating by parts using $u = x^n$ and $dv = e^{2x}dx$ yields the reduction formula

$I_n = x^n \cdot \tfrac{1}{2}e^{2x} - \int \tfrac{1}{2}e^{2x}\cdot(nx^{n-1})dx = \boxed{\tfrac{1}{2}x^n e^{2x} - \tfrac{n}{2}I_{n-1}}$ for $n \ge 1$.

Next, $I_0 = \tfrac{1}{2}e^{2x}$ implies

$I_1 = \tfrac{1}{2}xe^{2x} - \tfrac{1}{2}I_0 = \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x}$, and $I_2 = \tfrac{1}{2}x^2e^{2x} - I_1 = \tfrac{1}{2}x^2e^{2x} - \tfrac{1}{2}xe^{2x} + \tfrac{1}{4}e^{2x}$.

(d) *Find all first and second partial derivatives of* $f(x,y) = e^{x-y^3}$*.*

$f_x = e^{x-y^3}, \quad f_y = -3y^2e^{x-y^3}, \quad f_{xx} = e^{x-y^3}, \quad f_{xy} = -3y^2e^{x-y^3}, \quad f_{yy} = [-6y + 9y^4]e^{x-y^3}$.

(e) *Find the linearization of the function* $f(x,y) = e^{x-y^3}$ *about the point (0,0). (You may use the results of part (d).)*

$f(0,0) = 1, \quad f_x(0,0) = 1, \quad f_y(0,0) = 0.$

Answer: $f(0+h, 0+k) = 1 + 1\cdot h + 0\cdot k = 1 + h$.

(f) *Solve the differential equation* $x\dfrac{dy}{dx} + 3y = 3\ln x$ *(for* $x > 1$*),*

subject to the initial condition $y(1) = 2$.

To solve $y' + \frac{3}{x}y = 3x^{-1}\ln x$, find the integrating factor: $v = \exp\{\int \frac{3}{x}\,dx\} = x^3$.

So $(x^3 \cdot y)' = 3x^2 \ln x$. Therefore (using integration by parts with $u = \ln x$ and $v = 3x^2 dx$),

$x^3 \cdot y = \int 3x^2 \ln x\,dx = x^3 \ln x - \int x^3(1/x)dx = x^3 \ln x - \frac{1}{3}x^3 + C$ so $\boxed{y = \ln x - \frac{1}{3} + Cx^{-3}}$. The initial condition yields: $2 = 0 - \frac{1}{3} + C$ so $C = \frac{7}{3}$, so $\boxed{y = \ln x - \frac{1}{3} + \frac{7}{3}x^{-3}}$.

(g) *Evaluate the three determinants*

$$\begin{vmatrix} 1 & -2 & 7 \\ -2 & 1 & 5 \\ -2 & 1 & 5 \end{vmatrix}, \quad \begin{vmatrix} 1 & 3 & -2 \\ -2 & 4 & 1 \\ -2 & 1 & 1 \end{vmatrix}, \textit{ and } \begin{vmatrix} 1 & 3 & -2 & 7 \\ -2 & 4 & 1 & 5 \\ 0 & 3 & 0 & -1 \\ -2 & 1 & 1 & 5 \end{vmatrix}$$

Answers: $\boxed{0}$ (computed directly, or using the fact that the rows 2 and 3 are equal) and $\boxed{-9}$, and then, using the thrid row expansion, $-3 \cdot 0 - (-1) \cdot (-9) = \boxed{-9}$.

(h) *Prove that the determinant* $\begin{vmatrix} a_1 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & 0 & c_3 & 0 \\ d_1 & 0 & d_3 & d_4 \end{vmatrix} = a_1\, b_2\, c_3\, d_4.$

The prof is by direct evaluation. Use the first row expansion; for the resulting 3×3 determinant, use the first column expansion.

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2.(a) *A solid of revolution is obtained by rotating about the y-axis the area bounded between* $y = \dfrac{1}{x^3+x}$ *and the* x*-axis for* $0 \le x \le 4$. *Find the volume of the solid obtained.*

Cylindrical shell area: $2\pi x\left[\frac{1}{x^3+x}\right] = 2\pi\frac{1}{x^2+1}$.

$V = 2\pi \int_0^4 \frac{1}{x^2+1}\,dx = 2\pi \tan^{-1} x\Big|_0^4 = \boxed{2\pi\tan^{-1}4 \approx 8.330358068}$.

(b) *Find the arc-length of the curve* $y = (2x+1)^{3/2}$ *for* $0 \le x \le 1$.

$y'(x) = 2\dfrac{3}{2}(2x+1)^{1/2} = 3(2x+1)^{1/2}$. $\qquad \sqrt{1+y'^2} = \sqrt{1+9(2x+1)} = \sqrt{18x+10}$.

Arc-length (using the substitution $u = 18x+10$):

$$s = \int_0^1 \sqrt{18x+10}\,dx = \frac{1}{18}\int_{10}^{28}\sqrt{u}\,du = \frac{1}{18}\cdot\frac{2}{3}u^{3/2}\Big|_{10}^{28} = \frac{1}{27}[28^{3/2}-10^{3/2}] \approx 4.316270252.$$

(c) *Find the mass and the centre of mass of a rod with mass density* $\rho(x) = \sqrt{1-x^2}$ *for* $0 \le x \le 1$.

Mass (in view of the area of a quarter circle of radius 1 being $\frac{1}{4}\pi$):

$m = \int_0^1 \rho\,dx = \int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4}\pi \approx 0.7853981635.$

Moment (using $u = 1-x^2$):

$M = \int_0^1 x\rho\,dx = \int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_1^0 \sqrt{u}\,du = \frac{1}{3}u^{3/2}\Big|_0^1 = \frac{1}{3} \approx 0.3333333333$

Center of mass: $\boxed{\bar{x} = M/m = \frac{\frac{1}{3}}{\frac{1}{4}\pi} = \frac{4}{3\pi} \approx 0.4244131814}$.

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3.(a) *Find general solutions of the given differential equations:*

(i) $y'' - 3y' + 2y = 0,$ *(ii)* $y'' - 4y' + 4y = 0.$

(i) Roots: 2 and 1 so $y = C_1 e^{2x} + C_2 e^x$.

(ii) Roots: 2 and 2 so $y = C_1 e^{2x} + C_2 x e^{2x}$.

(b) *Find a particular solution for each of the given differential equations:*

(i) $y'' - 3y' + 2y = 2e^{2x} - 4,$

(ii) $y'' - 4y' + 4y = 2e^{2x} - 4.$

Then find the general solutions of these equations.

(i) Look for a particular solution in the form $y_p = A\,xe^{2x} + B$, which yields

$A\,e^{2x} + 2B = 2e^{2x} - 4$ so $y_p = 2\,xe^{2x} - 2$.

General solution: $y = 2\,xe^{2x} - 2 + C_1 e^{2x} + C_2 e^x$.

(ii) Look for a particular solution $y_p = A\,x^2 e^{2x} + B$, which yields

$2A\,e^{2x} + 4B = 2e^{2x} - 4x$ so $y_p = x^2 e^{2x} - 1$.

General solution: $y = x^2 e^{2x} - 1 + C_1 e^{2x} + C_2\,xe^{2x}$.

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4.(a) *Find the Taylor Series, up to and including quadratic terms, of* $z = f(x,y) = y\ \ln(x+y)$ *about the point* $(0,1)$.

Answer: $\boxed{f(h, 1+k) \approx h + k - \frac{1}{2}h^2 + \frac{1}{2}k^2}$.

$f_x = \frac{y}{x+y}, \quad f_y = \ln(x+y) + \frac{y}{x+y}, \quad f_{xx} = \frac{-y}{(x+y)^2}, \quad f_{xy} = \frac{1}{x+y} - \frac{y}{(x+y)^2} = \frac{x}{(x+y)^2},$

$f_{yy} = \frac{1}{x+y} + \frac{1\cdot(x+y) - y\cdot 1}{(x+y)^2} = \frac{2x+y}{(x+y)^2}.$

Hence, $f(0,1) = 0$,

$f_x(0,1) = 1, \quad f_y(0,1) = 1, \quad f_{xx}(0,1) = -1, \quad f_{xy}(0,1) = 0, \quad f_{yy}(0,1) = 1.$

(b) *It is known that the quantities* $z > 0$ *and* $t > 0$ *are related by the formula* $z = \alpha t^{\beta}$, *with some unknown constants* $\alpha > 0$ *and* β. *By writing this as* $\ln z = \beta \ln t + \ln \alpha$, *one can use the method of least squares to find the best-fit line relating* $\ln z$ *to* $\ln t$ *and hence find an approximation of the constants* α *and* β. *For the given data points*

$(t,\ z) = (2,\ 8),\ (4,\ 5),\ (6,\ 4),\ (8,\ 3),\ (10,\ 1),$

use this method to find an approximation of the constants α *and* β.

$n = 5$, $(\ln t,\ \ln z) \approx (0.6931471806,\ 2.079441542), (1.386294361,\ 1.609437912), (1.791759469,\ 1.386294361),$ $(2.079441542,\ 1.098612289), (2.302585093,\ 0)$.

$\sum_{k=1}^{5} \ln t_k \approx 8.253227646, \quad \sum_{k=1}^{5} (\ln t_k)^2 \approx 15.23864230, \quad \sum_{k=1}^{5} \ln z_k \approx 6.173786104, \quad \sum_{k=1}^{5} \ln t_k \cdot \ln z_k \approx 8.440919824.$

$$a \approx \frac{n \cdot (8.440919824) - (8.253227646) \cdot (6.173786104)}{n \cdot (15.23864230) - (8.253227646)^2} \approx \boxed{-1.083147346},$$

$$b \approx \frac{(6.173786104) - a \cdot (8.253227646)}{n} \approx 3.0226495446, \qquad \text{so } \alpha = e^b \approx \boxed{20.54565625} \text{ and}$$

$\beta = a \approx \boxed{-1.083147346}$.

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5 NOTE: For detailed evaluations, see the **Maple solutions** attached.

(a) *Find all solutions of each system of linear equations:*

$$\text{(i)}\quad \begin{aligned} x + 4y - 2z &= 4 \\ 3x + 11y - 5z &= 8 \\ -2x - 5y + z &= 4 \end{aligned} \quad ; \quad \text{(ii)}\quad \begin{aligned} x + 4y - 2z &= 4 \\ 3x + 11y - 5z &= 8 \\ -2x - 5y + 2z &= 4 \end{aligned}$$

(i) This system can be reduced to $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -12 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so, deleting the zero third row, and letting $z = t$, one gets $\boxed{x = -12 - 2t,\ y = 4 + t,\ z = t}$.

(ii) This system can be reduced to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{array}\right]$ so $\boxed{x = -12,\ y = 4,\ z = 0}$.

(b) *Find the inverse of the matrix*

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ -3 & -4 & 2 & -4 \\ 1 & 4 & 1 & -1 \\ 3 & 2 & 3 & 36 \end{bmatrix}.$$

From $\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 \\ -3 & -4 & 2 & -4 & 0 & 1 & 0 & 0 \\ 1 & 4 & 1 & -1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 3 & 36 & 0 & 0 & 0 & 1 \end{array}\right]$ get $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -244 & -49 & 68 & 10 \\ 0 & 1 & 0 & 0 & 195/2 & 39/2 & -27 & -4 \\ 0 & 0 & 1 & 0 & -121 & -24 & 34 & 5 \\ 0 & 0 & 0 & 1 & 25 & 5 & -7 & -1 \end{array}\right]$,

and then $A^{-1} = \begin{bmatrix} -244 & -49 & 68 & 10 \\ 195/2 & 39/2 & -27 & -4 \\ -121 & -24 & 34 & 5 \\ 25 & 5 & -7 & -1 \end{bmatrix}$.

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