MA4002 Final Exam Answers, Spring 2024

1.(a) *An object has acceleration* $a(t) = \dfrac{6}{(t+1)^4}$ *metres/second*^{2} *at time* t*. The initial velocity at time* $t = 0$ *is* $v = 1$ *metres/second. How far does it travel in the first* 9 *seconds?*

Velocity: $v(t) = 1 + \int_0^t 6(s+1)^{-4}\,ds = 1 - 2(s+1)^{-3}\Big|_0^t = 3 - 2(t+1)^{-3}$.

Distance $s = \int_0^9 v(t)\,dt = \big(3t + (t+1)^{-2}\big)\Big|_0^9 = 26.01$m so $\boxed{s = 26.01\text{ m}}$.

(b) *Consider the plane region bounded by the curves* $y = \cos x$ *and the* x*-axis for* $0 \le x \le 1$*. Find the volume of each of the* <u>two solids</u> *obtained by rotating this plane region* **(i)** *about the* x*-axis;* **(ii)** *about the* y*-axis.*

(i) The cross-sectional area: $\pi[(\cos x)^2 - (0)^2] = \frac{1}{2}\pi[1 + \cos(2x)]$.

$V = \frac{1}{2}\pi \int_0^1 [1 + \cos(2x)]\,dx = \frac{1}{2}\pi\big(x + \frac{1}{2}\sin(2x)\big)\Big|_0^1 = \boxed{\frac{1}{2}\pi + \frac{1}{4}\pi\sin(2) \approx 2.284956856.}$

(ii) Using cylindrical shells:

$V = \int_0^1 2\pi x\,[\cos x - 0]\,dx = 2\pi\big(x\sin x\big|_0^1 - \int_0^1 1\cdot\sin x\,dx\big) = 2\pi\big(x\sin x + \cos x\big)\Big|_0^1$

$= \boxed{2\pi(\sin(1) + \cos(1) - 1) \approx 2.398752333}.$

(c) *Obtain an iterative reduction formula for* $I_n = \displaystyle\int_1^e x^2(\ln x)^n dx$*, where* $n \ge 0$ *(Hint: integrate by parts.) Evaluate* I_0*. Then, using the reduction formula obtained, evaluate* I_1 *and* I_2*.*

Integrating by parts using $u = (\ln x)^n$ and $dv = x^2dx$ yields $du = n(\ln x)^{n-1}/x$ and $v = \frac{1}{3}x^3$, and the reduction formula for $n \ge 1$:

$I_n = (\ln x)^n \cdot \frac{1}{3}x^3\Big|_1^e - \int_1^e \frac{1}{3}x^3 \cdot n(\ln x)^{n-1}/x\,dx = \boxed{\frac{1}{3}e^3 - \frac{n}{3}I_{n-1}}$ for $n \ge 1$.

Next, $I_0 = \int_1^e x^2dx = \frac{1}{3}[e^3 - 1] \approx 6.361845636$ implies

$I_1 = \frac{1}{3}e^3 - \frac{1}{3}I_0 = \frac{1}{3}e^3 - \frac{1}{3}\cdot\frac{1}{3}[e^3 - 1] = \frac{2}{9}e^3 + \frac{1}{9} \approx 4.574563759$, and

$I_2 = \frac{1}{3}e^3 - \frac{2}{3}I_1 = \frac{1}{3}e^3 - \frac{2}{3}\cdot\left[\frac{2}{9}e^3 + \frac{1}{9}\right] = \frac{5}{27}e^3 - \frac{2}{27} \approx 3.645469800.$

(d) *Find all first and second partial derivatives of* $f(x,y) = \ln(x^2 - y)$*.*

$f_x = \frac{2x}{x^2-y}, \quad f_y = -\frac{1}{x^2-y}, \quad f_{xx} = 2\frac{(x^2-y) - x\cdot 2x}{(x^2-y)^2} = -2\frac{x^2+y}{(x^2-y)^2}, \quad f_{xy} = \frac{2x}{(x^2-y)^2}, \quad f_{yy} = -\frac{1}{(x^2-y)^2}.$

(e) *Find the linearization of the function* $f(x,y) = \ln(x^2 - y)$ *about the point (1,0). (You may use the results of part (d).)*

$f(1,0) = 0, \quad f_x(1,0) = 2, \quad f_y(1,0) = -1.$

Answer: $f(1+h, 0+k) = 0 + 2 \cdot h + (-1) \cdot k = 2h - k.$

(f) *Solve the differential equation* $x\dfrac{dy}{dx} - 3y = x^5 \cos(x^2+1)$ *(for* $x > 1$*), subject to the initial condition* $y(1) = 2$.

To solve $y' - \frac{3}{x}y = x^4 \cos(x^2+1)$, find the integrating factor: $v = \exp\{-\int \frac{3}{x}\,dx\} = x^{-3}$.

So $(x^{-3} \cdot y)' = x\cos(x^2+1)$. Therefore (using the substitution $u = x^2+1$ with $\frac{1}{2}du = x\,dx$), $x^{-3} \cdot y = \int x\cos(x^2+1)\,dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(x^2+1) + C$ so $\boxed{y = \frac{1}{2}x^3\sin(x^2+1) + Cx^3}$.

The initial condition yields: $2 = \frac{1}{2}\sin(1^2+1) + C$ so $C = 2 - \frac{1}{2}\sin 2 \approx 1.545351287$, so $\boxed{y = \frac{1}{2}x^3\sin(x^2+1) + (2 - \frac{1}{2}\sin 2)x^3}$.

(g) *Evaluate the three determinants*

$$\begin{vmatrix} 2 & 1 & 4 \\ 1 & 1 & -5 \\ 2 & -1 & 1 \end{vmatrix}, \quad \begin{vmatrix} 2 & -3 & 4 \\ 1 & 3 & -5 \\ 2 & -2 & 1 \end{vmatrix}, \textit{ and } \begin{vmatrix} 2 & -3 & 1 & 4 \\ 0 & 1 & 2 & 0 \\ 1 & 3 & 1 & -5 \\ 2 & -2 & -1 & 1 \end{vmatrix}.$$

Answers: $\boxed{-31}$ and $\boxed{-13}$,

and then, using the second row expansion, $-0 + 1 \cdot [-31] - 2 \cdot [-13] + 0 = \boxed{-5}$.

(h) *Evaluate the determinant* $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & 0 & 0 & 0 \\ c_1 & 0 & 0 & c_4 \\ d_1 & 0 & d_3 & d_4 \end{vmatrix}$.

Use, e.g., the second column expansion; for the resulting 3×3 determinant, use the first row expansion:

$$\det = -a_2 \begin{vmatrix} b_1 & 0 & 0 \\ c_1 & 0 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} = -a_2 \cdot b_1 \begin{vmatrix} 0 & c_4 \\ d_3 & d_4 \end{vmatrix} = -a_2 \cdot b_1 \cdot (-c_4 \cdot d_3) = a_2 b_1 c_4 d_3 .$$

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2.(a) *A solid of revolution is obtained by rotating about the y-axis the area bounded between* $y = \dfrac{1}{x^2 + 2x + 2}$ *and the x-axis for $0 \le x \le 2$. Find the volume of the solid obtained.*

Cylindrical shell area: $2\pi x \left[\frac{1}{x^2+2x+2}\right] = 2\pi \frac{x}{(x+1)^2+1}$. So, using $u = x + 1$, so $x = u - 1$,

$V = 2\pi \int_0^2 \frac{x}{(x+1)^2+1}\,dx = 2\pi \int_{u=1}^{u=3} \frac{u-1}{u^2+1}\,du = 2\pi\left(\frac{1}{2}\ln(u^2+1) - \tan^{-1} u\right)\Big|_1^3$

$= 2\pi\left(\frac{1}{2}(\ln 10 - \ln 2) - \tan^{-1} 3 + \tan^{-1} 1\right) = \boxed{2\pi\left(\frac{1}{2}\ln 5 - \tan^{-1} 3 + \frac{\pi}{4}\right) \approx 2.143014482}$.

(b) *Find the arc-length of the curve $y = (1 - x^2)^{1/2}$ for $0 \le x \le \dfrac{1}{2}$.*

$y'(x) = -x(1 - x^2)^{-1/2}$. $\qquad \sqrt{1 + y'^2} = \sqrt{1 + \dfrac{x^2}{1 - x^2}} = \dfrac{1}{\sqrt{1 - x^2}}$.

Arc-length $\quad s = \displaystyle\int_0^{1/2} \frac{1}{\sqrt{1 - x^2}}\,dx = \sin^{-1} x\Big|_0^{1/2} = \boxed{\sin^{-1}(\tfrac{1}{2}) \approx 0.5235987758}$.

(c) *Find the mass and the centre of mass of a rod with mass density $\rho(x) = \ln x$ for $1 \le x \le 2$.*

(NOTE that the left endpoint is at $x = 1$.)

Mass (using integration by parts with $u = \ln x$ and $v = x$):

$m = \int_1^2 \rho\,dx = \int_1^2 \ln x\,dx = \ln x \cdot x\Big|_1^2 - \int_1^2 x \cdot \frac{1}{x}\,dx = 2\ln 2 - x\Big|_1^2 = \boxed{2\ln 2 - 1 \approx 0.386294361}$.

Moment (using integration by parts with $u = \ln x$ and $v = \frac{1}{2}x^2$):

$M = \int_1^2 x\rho\,dx = \int_1^2 x\ln x\,dx = \ln x \cdot \frac{1}{2}x^2\Big|_1^2 - \int_1^2 \frac{1}{2}x^2 \cdot \frac{1}{x}\,dx = 2\ln 2 - \frac{1}{2} \cdot \frac{1}{2}x^2\Big|_1^2 = \boxed{2\ln 2 - \frac{3}{4} \approx 0.6362943610}$.

Center of mass: $\boxed{\bar{x} = M/m = \frac{2\ln 2 - \frac{3}{4}}{2\ln 2 - 1} \approx 1.647174863}$.

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3.(a) *Find general solutions of the given differential equations:*

(i) $y'' - 2y' + y = 0$, *(ii)* $y'' - 3y' = 0$.

(i) Roots: 1 and 1 so $y = C_1 e^x + C_2 x e^x$.

(ii) Roots: 0 and 3 so $y = C_1 + C_2 e^{3x}$.

(b) *Find a particular solution for each of the given differential equations:*

(i) $y'' - 2y' + y = 2e^x + 6$,

(ii) $y'' - 3y' = 2e^x + 6$.

Then find the general solutions of these equations.

(You may use the results of part (a).)

(i) Look for a particular solution in the form $y_p = A\,x^2 e^x + B$, which yields

$2A\,e^x + B = 2e^x + 6$ so $y_p = x^2 e^x + 6$.

General solution: $y = x^2 e^x + 6 + C_1 e^x + C_2 x e^x$.

(ii) Look for a particular solution $y_p = A\,e^x + B\,x$, which yields

$-2A\,e^x - 3B = 2e^x + 6$ so $y_p = -e^x - 2x$.

General solution: $y = -e^x - 2x + C_1 + C_2 e^{3x}$.

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4.(a) *Find the Taylor Series, up to and including quadratic terms,*

of $z = f(x, y) = x\,\cos(x - y^2)$ *about the point* $(0, 0)$.

Answer: $\boxed{f(h, k) \approx h}$.

$f_x = \cos(x - y^2) - x\,\sin(x - y^2), \quad f_y = 2xy\,\sin(x - y^2),$

$f_{xx} = -2\sin(x - y^2) - x\,\cos(x - y^2),$

$f_{xy} = 2y\,\sin(x - y^2) + 2xy\,\cos(x - y^2),$

$f_{yy} = 2x\,\sin(x - y^2) - 4xy^2\,\cos(x - y^2).$

Hence, $f(0, 0) = 0,$

$f_x(0, 0) = 1, \quad f_y(0, 0) = 0, \quad f_{xx}(0, 0) = 0, \quad f_{xy}(0, 0) = 0, \quad f_{yy}(0, 0) = 0.$

(b) *It is known that the quantities* $z > 0$ *and* $t > 0$ *are related by the formula* $z = \alpha t^\beta$, *with some unknown constants* $\alpha > 0$ *and* β. *By writing this as* $\ln z = \beta \ln t + \ln \alpha$, *one can use the method of least squares to find the best-fit line relating* $\ln z$ *to* $\ln t$ *and hence find an approximation of the constants* α *and* β. *For the given data points*

$$(t,\, z) = (1,\, 2),\ (2,\, 2),\ (3,\, 4),\ (4,\, 3),\ (5,\, 6),\ (6,\, 7),$$

use this method to find an approximation of the constants α *and* β.

$n = 6, \quad (\ln t,\, \ln z) \approx$

$(0,\, 0.6931471806), (0.6931471806,\, 0.6931471806), (1.098612289,\, 1.386294361), (1.386294361,\, 1.098612289),$

$(1.609437912,\, 1.791759469), (1.791759469,\, 1.945910149).$

$$\sum_{k=1}^{6} \ln t_k \approx 6.579251212, \quad \sum_{k=1}^{6} (\ln t_k)^2 \approx 9.409906419, \quad \sum_{k=1}^{6} \ln z_k \approx 7.608870630,$$

$$\sum_{k=1}^{6} \ln t_k \cdot \ln z_k \approx 9.896781610.$$

$$a \approx \frac{n \cdot (9.896781610) - (6.579251212) \cdot (7.608870630)}{n \cdot (9.409906419) - (6.579251212)^2} \approx \boxed{0.7075149747},$$

$$b \approx \frac{(7.608870630) - a \cdot (6.579251212)}{n} \approx 0.4923253125,$$

so $\alpha = e^b \approx \boxed{1.636116282}$ and $\beta = a \approx \boxed{0.7075149747}$.

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5 NOTE: For detailed evaluations, see the **Maple solutions** attached.

(a) *Find all solutions of each system of linear equations:*

$$(i)\quad \begin{aligned} x + 2y - 2z &= 1 \\ 2x + 5y - 5z &= 3 \\ 4x - y + z &= -9 \end{aligned} \quad ; \quad (ii)\quad \begin{aligned} x + 2y - 2z &= 1 \\ 2x + 5y - 5z &= 3 \\ 4x - y + z &= -5 \end{aligned}$$

(i) *This system can be reduced to* $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right]$ *or similar so (because of the final row)* $\boxed{\text{NO solutions}}$.

(ii) *This system can be reduced to* $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ *so, deleting the zero third row, and letting* $z = t$, *one gets* $\boxed{x = -1,\ y = 1 + t,\ z = t}$.

(b) *Find the inverse of the matrix*

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & -3 & -2 & -4 \\ -2 & 3 & 1 & -1 \\ 4 & -1 & -5 & 8 \end{bmatrix}.$$

From $\left[\begin{array}{cccc|cccc} 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 2 & -3 & -2 & -4 & 0 & 1 & 0 & 0 \\ -2 & 3 & 1 & -1 & 0 & 0 & 1 & 0 \\ 4 & -1 & -5 & 8 & 0 & 0 & 0 & 1 \end{array}\right]$ *get* $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 93 & -20 & 8 & -9 \\ 0 & 1 & 0 & 0 & 42 & -9 & 4 & -4 \\ 0 & 0 & 1 & 0 & 50 & -11 & 4 & -5 \\ 0 & 0 & 0 & 1 & -10 & 2 & -1 & 1 \end{array}\right]$,

and then $A^{-1} = \begin{bmatrix} 93 & -20 & 8 & -9 \\ 42 & -9 & 4 & -4 \\ 50 & -11 & 4 & -5 \\ -10 & 2 & -1 & 1 \end{bmatrix}$.

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