MA4002 Final Exam Answers, Spring 2025

1.(a) *An object has acceleration* $a(t) = \dfrac{3}{\sqrt{t+1}}$ *metres/second*^{2} *at time* t. *The initial velocity at time* $t = 0$ *is* $v = 1$ *metres/second. How far does it travel in the first* 8 *seconds?*

Velocity: $v(t) = 1 + \int_0^t \frac{3}{\sqrt{s+1}}\, ds = 1 + 6\sqrt{s+1}\Big|_0^t = -5 + 6\sqrt{t+1}$.

Distance $s = \int_0^8 v(t)\, dt = \left(-5t + 4(t+1)^{3/2}\right)\Big|_0^8 = -40 + 4(9^{3/2} - 1) = 64\text{m}$ so $\boxed{s = 64\,\text{m}}$.

(b) *Consider the plane region bounded by the curve* $y = \sqrt{x^2+4}$ *and the* x*-axis for* $0 \le x \le 2$. *Find the volume of each of the* <u>two solids</u> *obtained by rotating this plane region* **(i)** *about the* x*-axis;* **(ii)** *about the* y*-axis.*

(i) The cross-sectional area: $\pi[(\sqrt{x^2+4})^2] = \pi(x^2+4)$.

$V = \pi \int_0^2 (x^2+4)\, dx = \pi\left(\frac{1}{3}x^3 + 4x\right)\Big|_0^2 = \boxed{\frac{32}{3}\pi \approx 33.51032165.}$

(ii) Using cylindrical shells and then $u = x^2 + 4$:

$V = \int_0^2 2\pi x\,[\sqrt{x^2+4}]\, dx = \pi \int_{u=4}^8 u^{1/2}\, du = \pi\frac{2}{3}u^{3/2}\Big|_{u=4}^8 = \pi\frac{2}{3}(8^{3/2} - 8) = \boxed{\pi\frac{16}{3}(\sqrt{8}-1) \approx 30.63559054}.$

(c) *Obtain an iterative reduction formula for* $I_n = \displaystyle\int_1^e x^{-2}(\ln x)^n dx$, *where* $n \ge 0$ *(Hint: integrate by parts.) Evaluate* I_0. *Then, using the reduction formula obtained, evaluate* I_1 *and* I_2.

Integrating by parts using $u = (\ln x)^n$ and $dv = x^{-2}dx$ yields $du = n(\ln x)^{n-1}/x$ and $v = -x^{-1}$, and the reduction formula for $n \ge 1$:

$I_n = (\ln x)^n \cdot (-x^{-1})\Big|_1^e - \int_1^e (-x^{-1}) \cdot n(\ln x)^{n-1}/x\, dx = \boxed{-e^{-1} + nI_{n-1}}$ for $n \ge 1$.

Next, $I_0 = \int_1^e x^{-2}dx = -[e^{-1} - 1^{-1}] = 1 - e^{-1} \approx 0.6321205588$ implies

$I_1 = -e^{-1} + 1 \cdot I_0 = -e^{-1} + 1 \cdot [1 - e^{-1}] = 1 - 2e^{-1} \approx 0.2642411176$, and

$I_2 = -e^{-1} + 2I_1 = -e^{-1} + 2 \cdot \left[1 - 2e^{-1}\right] = 2 - 5e^{-1} \approx 0.160602794$.

(d) *Find all first and second partial derivatives of* $f(x,y) = \sqrt{x - y^2}$.

$f_x = \frac{1}{2\sqrt{x-y^2}}, \quad f_y = \frac{-y}{\sqrt{x-y^2}}, \quad f_{xx} = \frac{-1}{4(x-y^2)^{3/2}}, \quad f_{xy} = \frac{y}{2(x-y^2)^{3/2}}, \quad f_{yy} = -\frac{1\cdot\sqrt{x-y^2} - y\cdot f_y}{x-y^2} = -\frac{(x-y^2)+y^2}{(x-y^2)^{3/2}} = \frac{-x}{(x-y^2)^{3/2}}$.

(e) *Find the linearization of the function* $f(x,y) = \sqrt{x - y^2}$ *about the point* $(2,1)$. *(You may use the results of part (d).)*

$f(2,1) = 1, \quad f_x(2,1) = \frac{1}{2}, \quad f_y(2,1) = -1.$

Answer: $f(2+h, 1+k) = 1 + \frac{1}{2} \cdot h + (-1) \cdot k = 1 + \frac{1}{2}h - k.$

(f) *Solve the differential equation* $x\dfrac{dy}{dx} + 2y = 3x \ln x$ *(for* $x > 1$*), with the initial condition* $y(1) = \dfrac{2}{3}$.

To solve $y' + \frac{2}{x}y = 3 \ln x$, find the integrating factor: $v = \exp\{\int \frac{2}{x}\,dx\} = x^2$.

So $(x^2 \cdot y)' = 3x^2 \ln x$. Therefore $x^2 \cdot y = \int 3x^2 \ln x\,dx = x^3 \ln x - \int x^3 x^{-1} dx = x^3 \ln x - \frac{1}{3}x^3 + C$ (where we used an integration by parts with $u = \ln x$ and $v\,dv = 3x^2\,dx$, so $du = x^{-1}dx$ and $v = x^3$), so $\boxed{y = x \ln x - \frac{1}{3}x + Cx^{-2}}$. The initial condition yields: $\frac{2}{3} = 0 - \frac{1}{3} + C$ so $C = 1$, so $\boxed{y = x \ln x - \frac{1}{3}x + x^{-2}}$.

(g) *Evaluate the three determinants*

$$\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & -5 \\ 2 & 1 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 3 & -1 \\ -1 & 2 & 1 \\ 2 & -1 & 1 \end{vmatrix}, \textit{ and } \begin{vmatrix} 1 & 3 & -1 & 1 \\ -1 & 2 & 1 & -5 \\ 0 & 3 & 0 & -2 \\ 2 & -1 & 1 & 1 \end{vmatrix}.$$

Answers: $\boxed{12}$ and $\boxed{15}$,

and then, using the third row expansion, $0 - (3) \cdot [12] + 0 - (-2) \cdot [15] + 0 = \boxed{-6}$.

(h) *Evaluate the determinant* $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & 0 & 0 & 0 \\ c_1 & c_2 & 0 & 0 \\ d_1 & 0 & d_3 & d_4 \end{vmatrix}$.

Use, e.g., the second row expansion; for the resulting 3×3 determinant, use the second row expansion:

$$\det = -b_1 \begin{vmatrix} a_2 & a_3 & a_4 \\ c_2 & 0 & 0 \\ 0 & d_3 & d_4 \end{vmatrix} = (-b_1) \cdot (-c_2) \begin{vmatrix} a_3 & a_4 \\ d_3 & d_4 \end{vmatrix} = b_1\, c_2\, (a_3\, d_4 - a_4\, d_3)\,.$$

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2.(a) *A solid of revolution is obtained by rotating* *<u>about the x-axis</u>* *the area bounded between* $y = \dfrac{1}{\sqrt{x^2+3x+2}}$ *and the* x*-axis for* $0 \le x \le 1$. *Find the volume of the solid obtained.*

Cross-sectional area: $\pi\left[\frac{1}{\sqrt{x^2+3x+2}}\right]^2 = \pi\frac{1}{x^2+3x+2}$. So, using $x^2+3x+2 = (x+1)(x+2)$,

$V = \pi\int_0^1\left(\frac{1}{x+1} - \frac{1}{x+2}\right)dx = \pi(\ln|x+1| - \ln|x+2|)\Big|_0^1 = \pi([\ln 2 - \ln 1] - [\ln 3 - \ln 2])$

$\boxed{= \pi(2\ln 2 - \ln 3) \approx 0.9037798841}$.

(b) *Find the arc-length of the curve* $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ *for* $1 \le x \le 2$.

$y'(x) = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$ *and* $y'^2 = \left(\frac{1}{2}x^2 - \frac{1}{2}x^{-2}\right)^2 = \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4}x^{-4}$.

$1 + y'^2 = \frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4}x^{-4} = \left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right)^2$, *so* $\sqrt{1+y'^2} = \frac{1}{2}x^2 + \frac{1}{2}x^{-2}$.

Arc-length $s = \int_1^2\left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right)dx = \left(\frac{1}{6}x^3 - \frac{1}{2}x^{-1}\right)\Big|_1^2 = \frac{7}{6} - \frac{1}{2}\left(-\frac{1}{2}\right) = \boxed{\frac{17}{12} \approx 1.416666667}$.

(c) *Find the mass and the centre of mass of a rod with mass density* $\rho(x) = e^{-x}$ *for* $0 \le x \le 1$.

Mass:

$m = \int_0^1 \rho\, dx = \int_0^1 e^{-x}\, dx = -e^{-x}\Big|_0^1 = \boxed{1 - e^{-1} \approx 0.6321205588}$.

Moment (using integration by parts with $u = x$ *and* $v = -e^{-x}$*):*

$M = \int_0^1 x\rho\, dx = \int_0^1 x\, e^{-x}\, dx = x\,(-e^{-x})\Big|_0^1 - \int_0^1(-e^{-x})\, dx = (-xe^{-x} - e^{-x})\Big|_0^1 = -(x+1)e^{-x}\Big|_0^1 =$

$\boxed{1 - 2e^{-1} \approx 0.2642411176}$.

Center of mass: $\boxed{\bar{x} = M/m = \frac{1-2e^{-1}}{1-e^{-1}} \approx 0.4180232931}$.

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3.(a) *Find general solutions of the given differential equations:*

(i) $y'' + 2y' + 5y = 0,$ *(ii)* $y'' + 2y' + y = 0.$

(i) *Roots:* $-1 + 2i$ *and* $-1 - 2i$ *so* $y = e^{-x}(C_1 \cos(2x) + C_2 \sin(2x)).$

(ii) *Roots:* -1 *and* -1 *so* $y = C_1 e^{-x} + C_2 x e^{-x}.$

(b) *Find a particular solution for each of the given differential equations:*

(i) $y'' + 2y' + 5y = 4e^{-x} + 5,$

(ii) $y'' + 2y' + y = 4e^{-x} + 5.$

Then find the general solutions of these equations.

(You may use the results of part (a).)

(i) *Look for a particular solution in the form* $y_p = A\, e^{-x} + B$, *which yields*

$4A\, e^{-x} + 5B = 4e^{-x} + 5$ *so* $y_p = e^{-x} + 1.$

General solution: $y = e^{-x} + 1 + e^{-x}(C_1 \cos(2x) + C_2 \sin(2x)).$

(ii) *Look for a particular solution* $y_p = A\, x^2\, e^{-x} + B$, *which yields*

$2A\, e^{-x} + B = 4e^{-x} + 5$ *so* $y_p = 2x^2\, e^{-x} + 5.$

General solution: $y = 2x^2\, e^{-x} + 5 + C_1 e^{-x} + C_2 x e^{-x}.$

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4.(a) *Find the Taylor Series, up to and including quadratic terms, of* $z = f(x, y) = (x + 1)\, e^{xy}$ *about the point* $(0, 0)$.

Answer: $\boxed{f(h, k) \approx 1 + h + hk}$.

$f_x = 1 \cdot e^{xy} + y\,(x + 1)\, e^{xy}, \quad f_y = x(x + 1)\, e^{xy} = (x^2 + x)\, e^{xy}$,

$f_{xx} = y\, e^{xy} + y\, f_x = (2y + y^2(x + 1))e^{xy}$,

$f_{xy} = [(x^2 + x)\, e^{xy}]_x = (2x + 1)\, e^{xy} + (x^2 + x)y\, e^{xy}$,

$f_{yy} = x^2(x + 1)\, e^{xy}$.

So, $f(0, 0) = 1, \quad f_x(0, 0) = 1, \quad f_y(0, 0) = 0, \quad f_{xx}(0, 0) = 0, \quad f_{xy}(0, 0) = 1, \quad f_{yy}(0, 0) = 0.$

$f(h, k) \approx 1 + 1 \cdot h + 0 \cdot k + \frac{1}{2}(0 \cdot h^2 + 2 \cdot 1 \cdot hk + 0 \cdot k^2)$.

(b) *It is known that the quantities* $z > 0$ *and* $t > 0$ *are related by the formula* $z = \alpha t^\beta$, *with some unknown constants* $\alpha > 0$ *and* β. *By writing this as* $\ln z = \beta \ln t + \ln \alpha$, *one can use the method of least squares to find the best-fit line relating* $\ln z$ *to* $\ln t$ *and hence find an approximation of the constants* α *and* β. *For the given data points*

$(t,\ z) = (1,\ 5),\ (3,\ 3),\ (5,\ 2),\ (7,\ 1),\ (9,\ 1)$,

use this method to find an approximation of the constants α *and* β.

$n = 5, \quad (\ln t,\ \ln z) \approx$

$(0,\ 1.609437912),\ (1.098612289,\ 1.098612289),\ (1.609437912,\ 0.6931471806),\ (1.945910149,\ 0),$
$(2.197224578,\ 0)$.

$$\sum_{k=1}^{5} \ln t_k \approx 6.851184928, \quad \sum_{k=1}^{5} (\ln t_k)^2 \approx 12.41160151, \quad \sum_{k=1}^{5} \ln z_k \approx 3.401197382,$$

$$\sum_{k=1}^{5} \ln t_k \cdot \ln z_k \approx 2.322526313.$$

$$a \approx \frac{n \cdot (2.322526313) - (6.851184928) \cdot (3.401197382)}{n \cdot (12.41160151) - (6.851184928)^2} \approx \boxed{-0.7731589321},$$

$$b \approx \frac{(3.401197382) - a \cdot (6.851184928)}{n} \approx 1.739650441,$$

so $\alpha = e^b \approx \boxed{5.695352213}$ *and* $\beta = a \approx \boxed{-0.7731589321}$.

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5 *NOTE: For detailed evaluations, see the* ***Maple solutions*** *attached.*

(a) *Find all solutions of each system of linear equations:*

(i)
$$\begin{aligned} x + 2y - 2z + t &= 2 \\ 2x + 5y - 5z + t &= 6 \\ 4x - y + 2z + 9t &= -7 \\ x + 2y - 2z + 2t &= 0 \end{aligned}$$
; (ii)
$$\begin{aligned} x + 2y - 2z + t &= 2 \\ 2x + 5y - 5z + t &= 6 \\ 4x - y + 2z + 9t &= -7 \end{aligned}$$

(i) *This system reduces from* $\left[\begin{array}{cccc|c} 1 & 2 & -2 & 1 & 2 \\ 2 & 5 & -5 & 1 & 6 \\ 4 & -1 & 2 & 9 & -7 \\ 1 & 2 & -2 & 2 & 0 \end{array}\right]$ *to* $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$

so it has a $\boxed{\text{UNIQUE solution } x = 4,\ y = -5,\ z = -5,\ t = -2}$.

(ii) *This system can reduces from* $\left[\begin{array}{cccc|c} 1 & 2 & -2 & 1 & 2 \\ 2 & 5 & -5 & 1 & 6 \\ 4 & -1 & 2 & 9 & -7 \end{array}\right]$ *to* $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & -5 & 5 \\ 0 & 0 & 1 & -4 & 3 \end{array}\right]$ *so, letting* $t = s$ *(a free variable), one gets* $\boxed{x = -2 - 3s,\ y = 5 + 5s,\ z = 3 + 4s,\ t = s}$ *where* $s \in \mathbb{R}$.

(b) *Find the inverse of the matrix*

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 5 & 8 & -2 & 4 \\ 5 & 6 & 2 & 1 \\ 1 & 2 & -5 & 8 \end{bmatrix}.$$

From $\left[\begin{array}{cccc|cccc} 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\ 5 & 8 & -2 & 4 & 0 & 1 & 0 & 0 \\ 5 & 6 & 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & -5 & 8 & 0 & 0 & 0 & 1 \end{array}\right]$*, using elementary row operations, one gets:*

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 62 & -27 & 14 & 4 \\ 0 & 2 & 0 & 0 & -75 & 33 & -17 & -5 \\ 0 & 0 & 1 & 0 & -33 & 14 & -7 & -2 \\ 0 & 0 & 0 & 1 & -19 & 8 & -4 & -1 \end{array}\right],$$

and then $A^{-1} = \begin{bmatrix} 62 & -27 & 14 & 4 \\ -\frac{75}{2} & \frac{33}{2} & -\frac{17}{2} & -\frac{5}{2} \\ -33 & 14 & -7 & -2 \\ -19 & 8 & -4 & -1 \end{bmatrix}.$

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