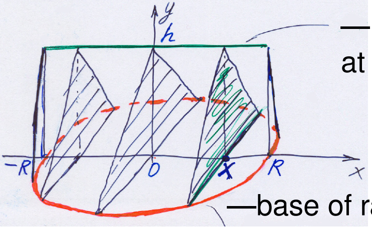
§17.1 Volumes by Slicing

Recall from §15.1: $V = \int_a^b A(x)\,dx$

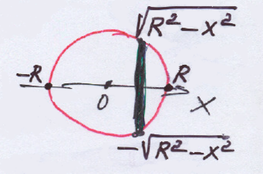
—for a general solid, where $A(x)$ is the cross-sectional area obtained by slicing the solid using planes perpendicular to the x-axis.

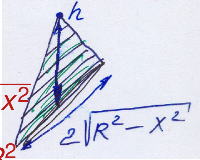
Final Example: A tent has a circular base of radius R and is supported by a horizontal bar held at height h. Find the volume of the tent.



—horizontal bar at height h

—base of radius R





Each cross-section: is a triangle of height h and base $2\sqrt{R^2-x^2}$, so the area is $A(x) = \frac{1}{2}h \cdot 2\sqrt{R^2-x^2}$

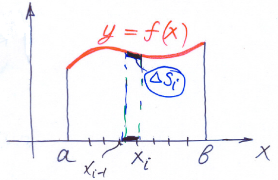
$$\Rightarrow\ V = \int_{-R}^{R} A(x)\,dx = \int_{-R}^{R} \tfrac{1}{2}h \cdot 2\sqrt{R^2-x^2}\,dx = \tfrac{1}{2}h \cdot \pi R^2.$$

§17.2 Arc-Length of a Curve

Assume:

$f(x)$ is smooth on $[a, b]$, i.e. $f'(x)$ exists and continuous on $[a, b]$.

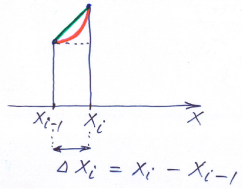
Consider a graph of a smooth $f(x)$—represented by a smooth curve:



What is the length of the curve??

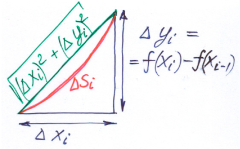
Solution: Partition $[a, b]$, as usual, into n subintervals.

On each subinterval:



our curve is approximately represented by the green straight line.

$$\Rightarrow \triangle s_i \approx \sqrt{(\triangle x_i)^2 + (\triangle y_i)^2}$$



Note: by the Mean-Value Theorem, $\triangle y_i = f(x_i) - f(x_{i-1}) = f'(c_i) \cdot \triangle x_i$,

where $c_i \in [x_{i-1}, x_i]$

$$\Rightarrow \ \triangle s_i \approx \sqrt{(\triangle x_i)^2 + \left(f'(c_i) \cdot \triangle x_i\right)^2} = \sqrt{1 + \left(f'(c_i)\right)^2} \cdot \triangle x_i.$$

For the total arc-length: $$s = \sum_{i=1}^{n} \triangle s_i \approx \underbrace{\sum_{i=1}^{n} \sqrt{1 + \left(f'(c_i)\right)^2} \cdot \triangle x_i}_{\text{Riemann sum}}$$

—here n is the number of subintervals.

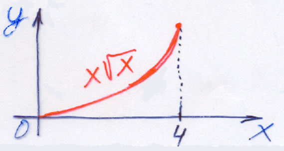
Let $n \to \infty$:

$$\boxed{s = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

—this is the arc-length of $y = f(x)$ between $x = a$ and $x = b$.

Examples:

1. Find the arc-length along $y = x\sqrt{x}$ for $0 \leqslant x \leqslant 4$.



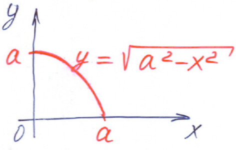
$$\frac{dy}{dx} = \left(x^{\frac{3}{2}}\right)' = \frac{3}{2}x^{\frac{1}{2}}, \qquad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{9}{4}x,$$

$$s = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

Substitution $u = 1 + \frac{9}{4}x$, $du = \frac{9}{4}\, dx$, $dx = \frac{4}{9}\, du$
with limits: $x = 0 \Rightarrow u = 1$, $x = 4 \Rightarrow u = 10$.

$$\Rightarrow\ s = \int_1^{10} \sqrt{u} \cdot \frac{4}{9}\, du = \frac{4}{9} \cdot \frac{2}{3}\, u^{\frac{3}{2}}\Big|_1^{10} = \frac{8}{27}\left(10^{\frac{3}{2}} - 1\right). \quad \square$$

2. Find the arc-length along $y = \sqrt{a^2 - x^2}$ for $0 \leqslant x \leqslant a$.
(Note: it's quarter of a circle!)



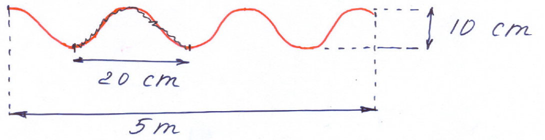
$$\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}},$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2},$$

$$s = \int_0^a \sqrt{\frac{a^2}{a^2 - x^2}}\, dx = a \sin^{-1}\left(\frac{x}{a}\right)\Big|_0^a = \frac{\pi a}{2}$$ —as expected! $\square$

3. Length of Corrugated Metal

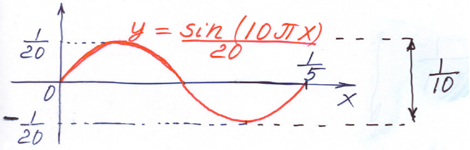
A sheet of metal is bent into the sinusoidal shape:



What length of sheet is required to make 5 m length of corrugated metal?

S: Note that $10\,\text{cm} = \frac{1}{10}\,\text{m}, \quad 20\,\text{cm} = \frac{1}{5}\,\text{m}.$

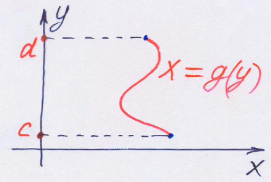
For a 5 m long panel, one needs 25 periods of the sheet:



$$\Rightarrow \; L = 25 \cdot \int_0^{\frac{1}{5}} \sqrt{1 + \left[\left(\frac{\sin(10\pi x)}{20}\right)'\right]^2}\,dx = 25 \int_0^{\frac{1}{5}} \sqrt{1 + \left[\frac{\pi}{2}\cos(10\pi x)\right]^2}\,dx$$

$\approx 7.32\,\text{m}$ —can be obtained using numerical integration. □

§17.3 Arc-Length of $x = g(y)$



$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

–an analogue of the previous formula (with $x \leftrightarrow y$ swap).

Note: $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$

—so the above formula can be formally obtained from the previous one...

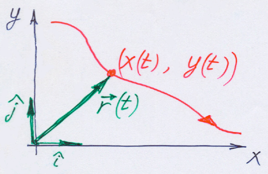
Example: Find the arc-length along $x = \frac{y^4}{32} + \frac{1}{y^2}$ for $1 \leqslant y \leqslant 2$.

S: $\frac{dx}{dy} = \frac{y^3}{8} - \frac{2}{y^3} \quad \Rightarrow \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \left(\frac{y^6}{64} - \frac{1}{2} + \frac{4}{y^6}\right)}$

$$= \sqrt{\frac{y^6}{64} + \frac{1}{2} + \frac{4}{y^6}} = \boxed{\sqrt{\left(\frac{y^3}{8} + \frac{2}{y^3}\right)^2}} = \frac{y^3}{8} + \frac{2}{y^3}.$$

$$\Rightarrow \; s = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2\, dy} = \int_1^2 \left(\frac{y^3}{8} + \frac{2}{y^3}\right) dy = \cdots = \frac{39}{32}. \quad \square$$

§18.1 A particle moves in the plane



—at time t its position is $\left(x(t),\, y(t)\right)$.

Alternatively, we can use a vector description:

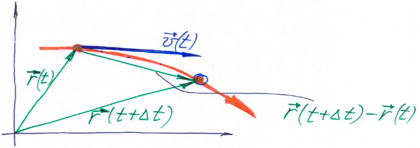
$$\boxed{\vec{r}(t) = x(t)\cdot\hat{i} + y(t)\cdot\hat{j}}.$$

. . . .

The velocity vector is $\boxed{\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\cdot\hat{i} + \frac{dy}{dt}\cdot\hat{j}}$.

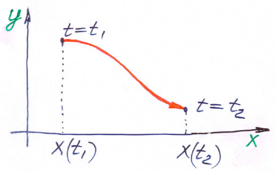
NOTE: this velocity vector has direction tangent to the path,

since $\vec{v}(t) = \frac{d\vec{r}}{dt} = \lim\limits_{\triangle t\to 0}\frac{\vec{r}(t+\triangle t)-\vec{r}(t)}{\triangle t}$:



§18.2 What distance does the particle travels

—between $t = t_1$ and $t = t_2$, given a particle trajectory $\vec{r}(t)$??



Solution:

forget about t for a moment
and recall the material of Lecture 17:
the arc-length of a curve $y(x)$

between $x = x(t_1)$ and $x = x(t_2)$ is given by $s = \int_{x(t_1)}^{x(t_2)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$

Now, use a substitution: $x = x(t)$:

$$\Rightarrow \; dx = \frac{dx}{dt}\, dt, \text{ with limits: } x = x(t_1) \Rightarrow t = t_1, \; x = x(t_2) \Rightarrow t = t_2.$$

Also use: $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ (by the Chain Rule) $\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$\Rightarrow \; s = \int_{t_1}^{t_2} \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2} \cdot \left(\frac{dx}{dt}\, dt\right) \quad \Rightarrow \quad \boxed{s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

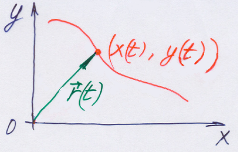
Remark: recall the velocity vector $\vec{v}(t) = \frac{dx}{dt} \cdot \hat{i} + \frac{dy}{dt} \cdot \hat{j}$

its magnitude: $\left|\vec{v}(t)\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$,

so rewrite our arc-length as $\boxed{s = \int_{t_1}^{t_2} \left|\vec{v}(t)\right| dt}$—i.e. the distance traveled is the integral, w.r.t. t of the velocity magnitude.

§18.3 Remark on Parametric Curves & the Arc-Length

A parametric curve in the plane is defined by 2 functions



$x = x(t)$ and $y = y(t)$,

or

the vector $\vec{r}(t) = x(t) \cdot \hat{i} + y(t) \cdot \hat{j}$

—similarly to movement of a particle (§18.1).

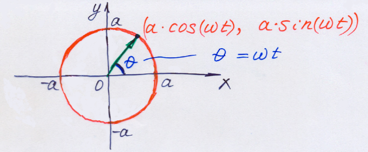
The difference: parameter t is NOT always time.

But mathematically: same thing, and same arc-length formula

$$\boxed{s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

§18.4 Examples

1. A particle has position $\vec{r}(t) = a\cos(\omega t)\cdot\hat{\imath} + a\sin(\omega t)\cdot\hat{\jmath}$ at time t (uniform circular motion):



Find the distance traveled between $t = 0$ and $t = T$.

S: $\vec{v}(t) = \frac{d\vec{r}}{dt} = -a\omega\sin(\omega t)\cdot\hat{\imath} + a\omega\cos(\omega t)\cdot\hat{\jmath}$

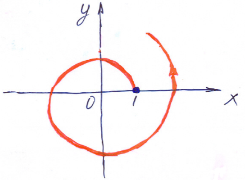
$$|\vec{v}| = \sqrt{(a\omega)^2\sin^2(\omega t) + (a\omega)^2\cos^2(\omega t)} = \sqrt{(a\omega)^2} = a\omega$$

$$\Rightarrow \quad s = \int_0^T |\vec{v}|\,dt = \int_0^T a\omega\,dt = a\omega T. \quad \square$$

NOTE: $s = a\theta$,
where $\theta = \omega T$ is the angle traveled between $t = 0$ and $t = T$.

Note also that ω is called the angular speed.

2. Find the length of the outward spiral $\vec{r}(t) = e^t\left(\cos t \cdot \hat{i} + \sin t \cdot \hat{j}\right)$ for $0 \leqslant t \leqslant T$.



S: $\vec{v}(t) = \frac{d\vec{r}}{dt} = \left(e^t \cos t - e^t \sin t\right) \cdot \hat{i} + \left(e^t \cdot \sin t + e^t \cos t\right) \hat{j}$

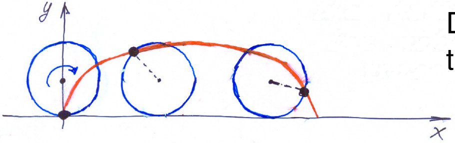
$$\left|\vec{v}\right|^2 = e^{2t}\left(\cos^2 t - \underline{2\cos t \sin t} + \sin^2 t\right) + e^{2t}\left(\sin^2 t + \underline{2\sin t \cos t} + \cos^2 t\right)$$
$$= 2e^{2t}$$

(where we used $\cos^2 t + \sin^2 t = 1$).

$$\Rightarrow \left|\vec{v}\right| = \sqrt{2}\, e^t$$

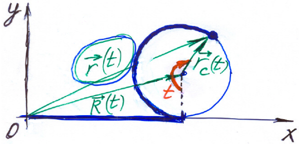
$$\Rightarrow s = \int_0^T \left|\vec{v}\right| dt = \int_0^T \sqrt{2}\, e^t\, dt = \sqrt{2}\, e^t\Big|_0^T = \sqrt{2}\left(e^T - 1\right). \quad \square$$

3. A cycloid: a circular wheel of radius a rolls along a straight line. Fix the point at the bottom of the wheel and trace its path.



Determine the length of the curve through one full revolution.

S: What is $\vec{r}(t)$?



We introduce parameter t = number of radians that the wheal has rolled through. (NOTE: t is not time here!)

Observe: $\boxed{\vec{r}(t) = \vec{R}(t) + \vec{r}_c(t)}$.

–After rolling through t radians: $\vec{R}(t) = (at, a) = at \cdot \hat{\imath} + a \cdot \hat{\jmath}$.

–Relative to the centre of the wheal, our fixed point goes along the circle, so has position: $\vec{r}_c(t) = (-a \sin t, -a \cos t)$.

Hence, $\vec{r}(t) = \vec{R}(t) + \vec{r}_c(t) = (at - a \sin t)\,\hat{\imath} + (a - a \cos t)\,\hat{\jmath}$.

Remark: we have shown that a parametric form of the cycloid:

$$\boxed{\begin{aligned} x &= at - a\sin t \\ y &= a - a\cos t \end{aligned}}$$

Back to our Solution:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (a - a\cos t)\,\hat{i} + (a\sin t)\,\hat{j},$$

$$\left|\vec{v}(t)\right|^2 = \underline{a^2}(1 - 2\cos t + \underbrace{\underline{\cos^2 t}) + \underline{a^2\ \sin^2 t}}_{=a^2\cdot 1} = a^2(2 - 2\cos t) = 4a^2\sin^2\left(\tfrac{t}{2}\right)$$

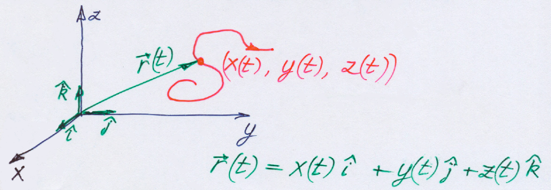
$$\Rightarrow\ \left|\vec{v}(t)\right| = 2a\left|\sin\left(\tfrac{t}{2}\right)\right| \qquad \Rightarrow\ s = \int_0^{2\pi} 2a\left|\sin\left(\tfrac{t}{2}\right)\right|\,dt$$

—here we integrate on $[0, 2\pi]$ as at the start $t = 0$, and after one full revolution $t = 2\pi$.

Note: $\left|\sin\left(\frac{t}{2}\right)\right| = \sin\left(\frac{t}{2}\right)$ for all $t \in [0, 2\pi]$ (as there $\sin\left(\frac{t}{2}\right) \geqslant 0$),

$$\Rightarrow\ s = 2a\int_0^{2\pi}\sin\left(\tfrac{t}{2}\right)dt = -4a\cos\left(\tfrac{t}{2}\right)\Big|_0^{2\pi} = -4a(-1-1) = 8a. \quad \square$$

Lecture 19 §19.1 Arc-Length: Generalization to 3D (Three Dimensions)

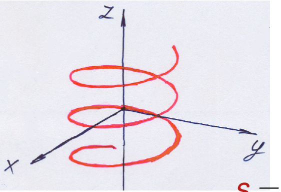


–A particle moves in space

$\Rightarrow$ The velocity vector is $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\cdot\hat{i} + \frac{dy}{dt}\cdot\hat{j} + \frac{dz}{dt}\cdot\hat{k}$

$$\boxed{s = \int_{t_1}^{t_2} |\vec{v}(t)|\, dt = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt}$$

Example: Spiral Helix



$\vec{r}(t) = a\cos t\cdot\hat{i} + a\sin t\cdot\hat{j} + b\,t\cdot\hat{k}$.

Find the arc-length for $0 \leqslant t \leqslant T$.

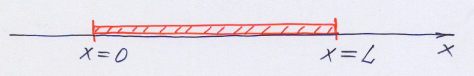
S: $\vec{v}(t) = \frac{d\vec{r}}{dt} = -a\sin t\cdot\hat{i} + a\cos t\cdot\hat{j} + b\cdot\hat{k}$,

$\left|\vec{v}(t)\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$,

$s = \int_0^T \left|\vec{v}(t)\right| dt = \int_0^T \sqrt{a^2+b^2}\, dt = \sqrt{a^2+b^2}\cdot T$. □

§19.2 Wires and Thin Rods: Density and Mass

Consider a thin rod (wire) of length L lying along the x-axis with one end at $x = 0$:

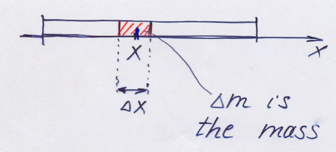


Density:

if a rod is made of homogeneous material, i.e. has constant density, then the density is $\boxed{\rho = \frac{m}{L}}$ —density per unit length, or line density, where m is the total mass, and L is the length.

Remark:

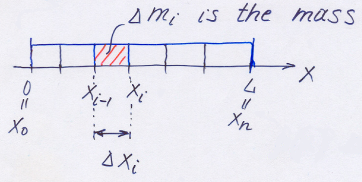
if a rod is NOT homogeneous, its density is not constant, so instead



$$\rho(x) \approx \frac{\triangle m}{\triangle x}.$$

In fact, $\boxed{\rho(x) = \lim_{\triangle x \to 0} \frac{\triangle m}{\triangle x}}$

Mass of an Inhomogeneous Rod:



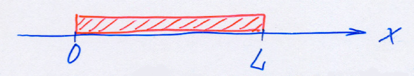
If $\rho(x)$ is a variable density,

$\Rightarrow\ \triangle m_i \approx \rho(x_i)\cdot\triangle x_i$.

Total Mass: $m = \sum_{i=1}^{n} \triangle m_i \approx \underbrace{\sum_{i=1}^{n} \rho(x_i)\cdot\triangle x_i}_{\text{Riemann Sum}}$

Let the number of subintervals $n\to\infty$: $\boxed{m = \int_0^L \rho(x)\,dx}$

Example:

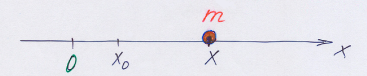


A rod of variable composition has density $\rho(x) = k\,x$. Find its mass.

S: We have $m = \int_0^L \rho(x)\,dx = \int_0^L k\,x\,dx = k\,\frac{x^2}{2}\Big|_0^L = \frac{k\,L^2}{2}$. $\square$

§19.3 Wires & Thin Rods: Moments and Centre of Mass

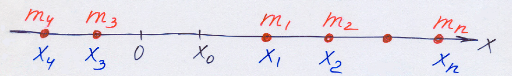
Recall a few Basic Definitions:



(1) A mass m, located at position x on the x-axis, is said to have moment $\boxed{x\, m}$ about the point 0,

and moment $\boxed{(x - x_0)\, m}$ about the point x_0.

. . . .



(2) If several masses $m_1, m_2, m_3, \cdots, m_n$ are located at $x_1, x_2, x_3, \cdots, x_n$, then the total moment of the system of masses about x_0 is the sum

$$\boxed{M_{x_0} = \sum_{i=1}^{n} (x_i - x_0)\, m_i}$$

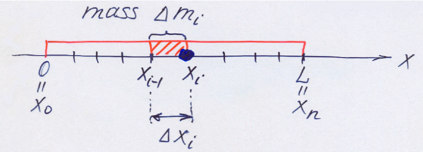
—measures the tendency of the system to rotate about x_0;

with a particular case $\boxed{M_0 = \sum_{i=1}^{n} x_i\, m_i}$.

(3) The centre of mass of the system of masses is the point $\bar{x}$ about which the total moment of the system is zero: $\boxed{M_{\bar{x}} = 0}$.

This definition $\Leftrightarrow$ $0 = \sum_{i=1}^{n} (x_i - \bar{x})\, m_i = \sum_{i=1}^{n} x_i\, m_i - \bar{x} \sum_{i=1}^{n} m_i = M_0 - \bar{x} \cdot m$,

$$\Rightarrow \boxed{\bar{x} = \frac{\sum_{i=1}^{n} x_i\, m_i}{\sum_{i=1}^{n} m_i} = \frac{M_0}{m}}$$



—Consider a thin rod of variable density $\rho(x)$.

Total moment about $x = 0$: $M_{x=0} \approx \sum_{i=1}^{n} x_i\, \triangle m_i \approx \underbrace{\sum_{i=1}^{n} x_i\, \rho(x_i)\, \triangle x_i}_{\text{Riemann Sum}}$

Let $n \to \infty$:

$\Rightarrow \boxed{M_0 = \int_0^L x\, \rho(x)\, dx}$ —total moment of the rod about $x = 0$.

Combine this with: $\boxed{m = \int_0^L \rho(x)\, dx}$ $\Rightarrow \boxed{\bar{x} = \frac{M_0}{m} = \frac{\int_0^L x\, \rho(x)\, dx}{\int_0^L \rho(x)\, dx}}$

Examples:

1. A rod lying along $[0,4]$ has variable density $4+x$ per unit length. Find the mass and the centre of mass of this rod.

   S: $\rho(x) = 4 + x \quad \Rightarrow \quad m = \int_0^4 \rho(x)\,dx = \int_0^4 (4+x)\,dx = 24.$

   $M_{x=0} = \int_0^4 x\,\rho(x)\,dx = \int_0^4 x\,(4+x)\,dx = \int_0^4 (4x + x^2)\,dx = \frac{160}{3}.$

   $\Rightarrow \bar{x} = \dfrac{M_{x=0}}{m} = \dfrac{\frac{160}{3}}{24} = \frac{20}{9}.$ Answer: $m = 24$ and $\bar{x} = \frac{20}{9}$.

2. A rod lying along $[0,2]$ has linear density $\rho(x) = a + b\,x$. The total mass is 8, and the centre of mass is $\frac{5}{6}$. Find a and b.

S: $\rho(x) = a + b\,x, \quad m = 8, \quad \bar{x} = \frac{5}{6}$

$$\Rightarrow 8 = \int_0^2 \rho(x)\,dx = \int_0^2 (a + b\,x)\,dx = 2a + 2b.$$

Also $\bar{x} = \dfrac{M_{x=0}}{m} \Rightarrow M_{x=0} = m \cdot \bar{x} = 8 \cdot \frac{5}{6} = \frac{20}{3}$

$$\Rightarrow \frac{20}{3} = \int_0^2 x\,\rho(x)\,dx = \int_0^2 x\,(a + b\,x)\,dx = \int_0^2 (a\,x + b\,x^2)\,dx$$

$$= \left(a\,\frac{x^2}{2} + b\,\frac{x^3}{3}\right)\Big|_0^2 = 2a + \frac{8}{3}b.$$

So we get: $\left.\begin{aligned} a + b &= 4 \\ 3a + 4b &= 10 \end{aligned}\right\} \Rightarrow \boxed{b = -2, \quad a = 6.}$

3. A rod of length L has mass density $\rho(x) = 2 + \sin\left(\frac{\pi x}{2L}\right)$ per unit length. Find the mass and the centre of mass of this rod.

S: $m = \int_0^L \rho(x)\,dx = \int_0^L \left[2 + \sin\left(\frac{\pi x}{2L}\right)\right] dx = \left[2x - \frac{2L}{\pi}\cos\left(\frac{\pi x}{2L}\right)\right]\Big|_0^L$

$$= \left[2L - \frac{2L}{\pi}\underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0}\right] - \left[0 - \frac{2L}{\pi}\underbrace{\cos(0)}_{=1}\right] \Rightarrow \boxed{m = 2L\left(1 + \frac{1}{\pi}\right)}.$$

$$M_{x=0} = \int_0^L x\,\rho(x)\,dx = \int_0^L x\left[2 + \sin\left(\frac{\pi x}{2L}\right)\right] dx$$

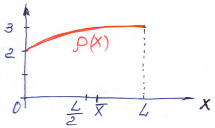
$$= \int_0^L 2x\,dx + \underbrace{\int_0^L \underbrace{x}_{u}\,\underbrace{\sin\left(\frac{\pi x}{2L}\right)dx}_{dv}}_{\text{by parts: } = uv - \int v\,du}$$

Here:
$du = dx$,
$v = -\frac{2L}{\pi}\cos\left(\frac{\pi x}{2L}\right)$

$$= x^2\Big|_0^L + x\left[-\frac{2L}{\pi}\cos\left(\frac{\pi x}{2L}\right)\right]\Big|_0^L - \int_0^L\left[-\frac{2L}{\pi}\cos\left(\frac{\pi x}{2L}\right)\right]dx$$

$$= L^2 + 0 + \frac{2L}{\pi}\cdot\frac{2L}{\pi}\sin\left(\frac{\pi x}{2L}\right)\Big|_0^L = L^2 + \frac{4L^2}{\pi^2}\left[\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} - \underbrace{\sin(0)}_{=0}\right] = L^2\left(1 + \frac{4}{\pi^2}\right).$$

$$\Rightarrow \bar{x} = \frac{M_{x=0}}{m} = \frac{L^2\left(1 + \frac{4}{\pi^2}\right)}{2L\left(1 + \frac{1}{\pi}\right)} \Rightarrow \boxed{\bar{x} = \frac{L\left(1 + \frac{4}{\pi^2}\right)}{2\left(1 + \frac{1}{\pi}\right)}} \approx .533\,L.$$



Definition

A differential equation is an equation that involves one or more derivatives of (an) unknown function(s).

(A) An Ordinary Differential Equation (ODE) involves derivatives w.r.t. one variable. E.g., $y\,\frac{dy}{dx} - x = 0$, where $y(x)$ is an unknown function.

A Partial Differential Equation (PDE) involves partial derivatives of an unknown function w.r.t. more than one variable.

E.g., $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, where $u(x,t)$ is an unknown function.

(B) The order of a differential equation is the order of the highest-order derivative in the equation.

E.g., $y\,\frac{dy}{dx} = x$ is a first-order ODE;

$x^2\,\frac{d^2 y}{dx^2} + x\,\frac{dy}{dx} + 2 = 0$ is a second-order ODE;

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$ is a second-order PDE.

(C) Linear DEs:

An *n*-th order linear ODE has the form:

$$a_n(x)\,\frac{d^n y}{dx^n} + a_{n-1}(x)\,\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2(x)\,\frac{d^2 y}{dx^2} + a_1(x)\,\frac{dy}{dx} + a_0(x)\,y(x) = f(x)$$

E.g., $x^2\,y'' + y' = \sin x$ is a linear second-order ODE;

$y^2\,y'' + y' = \sin x$ is a nonlinear ODE.

If $\boxed{f(x) = 0}$, it is called a homogeneous linear ODE.

E.g., $y'' + x\,y' + y = 0$ is homogeneous;

$y'' + x\,y' + x = 0$ is nonhomogeneous

(here $f(x) = -x$).

§20.2 Equations of Growth and Decay

Perhaps the best-known DE is $\boxed{\frac{dy}{dx} = k\,y}$ $(*)$

—the rate $\frac{dy}{dx}$ of change of $y(x)$ is proportional to the current value of y.

. . . .

To solve this: –Rewrite as $\boxed{\frac{1}{y} \cdot \frac{dy}{dx} = k}$, where $y = y(x)$.

–Integrate w.r.t. x: $\int^x \frac{1}{y} \cdot \frac{dy}{dx} \cdot dx = \int^x k \cdot dx + C$.

–Make a substitution $y = y(x)$ with $\frac{dy}{dx} \cdot dx = dy$

$$\Rightarrow \underbrace{\int^y \frac{dy}{y}}_{\ln|y|} = \underbrace{\int^x k \cdot dx}_{k\,x} + C \qquad (**)$$

$$\Rightarrow \ln|y| = kx + C \quad \Rightarrow |y| = e^{kx+C} = e^C\,e^{kx} = C_1\,e^{kx},$$

where $C_1 = e^C > 0 \quad \Rightarrow \; y = \pm C_1\,e^{kx}$,

$\Rightarrow \boxed{y = C\,e^{kx}}$ is a solution of $(*)$ for any constant C,

including $C = 0$.

Note: here $C = 0$ gives a solution $y = 0$ for all x

—check that it is indeed a particular solution of $(*)$.

Remark: $(**)$ can be obtained directly from $(*)$ as follows:

—Formally separate variables: $\underbrace{\frac{dy}{y}}_{\text{only } y} = \underbrace{k\,dx}_{\text{only } x}$

—Integrate: $\int^y \frac{dy}{y} = \int^x k\,dx + C$—so we again get $(**)$...

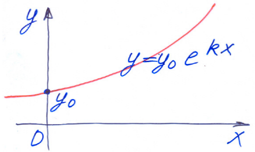
NOTE: $(*)$ is an example of a DE with separable variables.

If in addition to $(*)$, we know that $\boxed{y = y_0 \text{ when } x = 0}$ $(***)$
(this is called an initial condition),

then $y_0 = C\,e^{k\cdot 0} = C \cdot 1 = C$

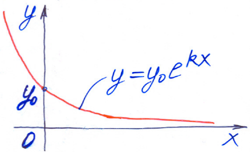
$\Rightarrow \boxed{y = y_0\,e^{kx}}$ is the unique solution of $(*)$, $(***)$.

(i) $y_0 > 0,\ k > 0$:



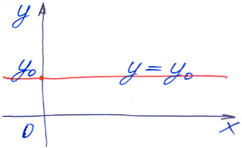
—exponential growth

(ii) $y_0 > 0,\ k < 0$:



—exponential decay

(iii) $y_0 > 0,\ k = 0$:



—constant

Examples:

1. Population Growth: A colony of bacteria increases in size at about 1% per minute. How long does it take to double in size?

   S: Let t = time (min); $y = y(t)$ = size of population at time t.

   For $y(t + \triangle t) - y(t)$ = the number of bacteria added during a small time interval $\triangle t$, one has

$$y(t + \triangle t) - y(t) \approx \underbrace{(.01) \cdot y(t)}_{\approx 1\% \text{ of the population}} \cdot \underbrace{\triangle t}_{\text{time}} \quad \Rightarrow \quad \frac{y(t+\triangle t) - y(t)}{\triangle t} \approx (.01) \cdot y(t)$$

Let $\triangle t \to 0$: $\boxed{\frac{dy}{dt} = (.01) \cdot y}$ $\Rightarrow$ Solution is $\boxed{y(t) = y_0\, e^{.01\, t}}$, where $y_0 = y(0)$ = initial size of the population.

   Back to our question:

   as $y = 2y_0$ at $t = T = ??$, so $2y_0 = y(T) = y_0\, e^{.01\, T}$,

   so $2 = e^{.01\, T}$, so $\ln 2 = .01\, T$, so $T = 100 \ln 2 \approx 69.3$(min). □

2. Radioactive Decay: Radioactive elements decay at a rate proportional to the number of radioactive elements present.

S: (Similarly to the previous example), we model this as

$$\boxed{\frac{dy}{dt} = -\lambda \cdot y}, \text{ with some } \lambda > 0,$$

$$\Rightarrow \text{The solution is } \boxed{y(t) = y_0\, e^{-\lambda\, t}}.$$

Here t = time, and $y(t)$ = the number of radioactive elements at time t.

Half-Life: The time $T_{1/2}$ it takes for half of the initial amount to decay is called half-life of the element.

Its relation to λ is obtained from: $\frac{1}{2}\, y_0 = y_0\, e^{-\lambda\, T_{1/2}}$

$$\Rightarrow \frac{1}{2} = e^{-\lambda\, T_{1/2}} \Rightarrow \underbrace{\ln\left(\frac{1}{2}\right)}_{=-\ln 2} = -\lambda\, T_{1/2} \Rightarrow \boxed{T_{1/2} = \frac{\ln 2}{\lambda}} \text{—the half-life}$$

Note also that $\boxed{\lambda = \frac{\ln 2}{T_{1/2}}}$

3. Radiocarbon Dating:

C-14 decays into C-12. Note that C-14 has a half-life of 5700 years. Find the age of a sample in which 20% of the C-14 decayed.

S: We have $\underbrace{T_{1/2}}_{5\,700} = \frac{\ln 2}{\lambda}, \quad \Rightarrow \lambda = \dfrac{\ln 2}{T_{1/2}} = \dfrac{\ln 2}{5\,700} \approx 1.216 \cdot 10^{-4}$.

We look for $t = T$ such that $\underbrace{0.8\, y_0}_{\text{80\% of the original amount of C-14}} = \underbrace{y_0\, e^{-\lambda T}}_{=y(T)}$

$$\Rightarrow \; e^{-\lambda T} = 0.8 \quad \Rightarrow \; -\lambda T = \ln(0.8)$$

$$\Rightarrow \; T = \frac{-\ln(0.8)}{\lambda} \approx \frac{.2231}{1.216 \cdot 10^{-4}} \approx 1\,800 \text{ years.} \quad \square$$

Lecture 21

Consider First-Order ODE of the form $\boxed{\frac{dy}{dx} = f(x,y)}$ $(*)$

NOTE:

The general solution of this ODE involves an arbitrary constant.
To single out a unique solution from those given by the general solution formula, we specify y at some initial value: $\boxed{y(x_0) = y_0}$ $(**)$

(this is called an initial condition);

$(*)$, $(**)$ is called an initial-value problem (IVP).

We shall consider a few special cases:

§21.1 Simplest Case $\boxed{\frac{dy}{dx} = f(x)}$

$$\Rightarrow \underbrace{\int_{x_0}^{x} \frac{dy}{dt}\,dt}_{=y(t)\big|_{x_0}^{x}=y(x)-y(x_0)=y(x)-y_0} = \int_{x_0}^{x} f(t)\,dt \quad \Rightarrow \quad \boxed{y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt}$$

—provided we can integrate f...

Remark: If y_0 is not specified, then $\boxed{y(x) = C + \int_{x_0}^{x} f(t)\,dt}$

where C is an arbitrary constant.

Example: $\frac{dy}{dx} = \cos x.$ (1)

S: $y(x) = \int \cos x \, dx = \sin x + C$ = general solution of ODE (1).

Add an initial condition, e.g., $y(0) = 2.$ (2)

Then $\underbrace{y(0)}_{=2} = \underbrace{\sin 0 + C}_{=C} \Rightarrow C = 2 \Rightarrow y(x) = 2 + \sin x$ is a unique solution of IVP (1), (2).

§21.2 First-Order ODEs with Separable Variables

$$\frac{dy}{dx} = f(x)\, g(y) \quad (3)$$
$$y(x_0) = y_0 \quad (4)$$

Formal Solution of IVP (3), (4):

—Separate variables: $\underbrace{\frac{dy}{g(y)}}_{\text{only } y} = \underbrace{f(x)\, dx}_{\text{only } x}$

—Integrate formally: $\int_{y_0}^{y} \frac{ds}{g(s)} = \int_{x_0}^{x} f(t)\, dt$ ...

Explanation/Justification: –Rewrite (3) as $\frac{1}{g(y)} \cdot \frac{dy}{dx} = f(x)$.

–Integrate w.r.t. x: $\int_{x_0}^{x} \frac{1}{g(y(t))} \cdot \frac{dy}{dt} \cdot dt = \int_{x_0}^{x} f(t) \cdot dt.$

–Make a substitution $s = y(t)$ with $\frac{dy}{dt} \cdot dt = ds$ and limits

$$\left.\begin{array}{l} t = x_0 \Rightarrow s = y(x_0) = y_0 \\ t = x \Rightarrow s = y(x) = y \end{array}\right\} \Rightarrow \int_{y_0}^{y} \frac{1}{g(s)}\, ds = \int_{x_0}^{x} f(t) \cdot dt$$

—the formula in our formal solution!

Similarly, Formal Solution of DE (3) only:

—Separate variables as before: $\underbrace{\frac{dy}{g(y)}}_{\text{only } y} = \underbrace{f(x)\,dx}_{\text{only } x}$

—Integrate formally: $\int^y \frac{dy}{g(y)} = \int^x f(x)\,dx + C$ ...

Examples: ....

1. Solve the IVP: $\frac{dy}{dx} = x^2 y^3$ and $y(1) = 3$.

S: –Separate variables: $\frac{dy}{y^3} = x^2\,dx$.

–Integrate: $\int \frac{dy}{y^3} = \int x^2\,dx + C. \;\Rightarrow\; -\frac{1}{2y^2} = \frac{x^3}{3} + C.$

–Use the initial condition to find C: $-\frac{1}{2\cdot 3^2} = \frac{1^3}{3} + C \;\Rightarrow\; C = -\frac{7}{18}$.

–Now $-\frac{1}{2y^2} = \frac{x^3}{3} - \frac{7}{18} \;\Rightarrow\; -2y^2 = \frac{1}{\frac{x^3}{3} - \frac{7}{18}} = \frac{18}{6x^3-7} \;\Rightarrow\; y^2 = \frac{9}{7-6x^3}$.

–Now it seems that we have 2 solutions: $y = +\frac{3}{\sqrt{7-6x^3}}$ and $y = -\frac{3}{\sqrt{7-6x^3}}$.

Recall the initial condition: $3 = +\frac{3}{\sqrt{7-6\cdot 1^3}}$—true; $3 = -\frac{3}{\sqrt{7-6\cdot 1^3}}$—false.

Answer: unique solution $\boxed{y = \frac{3}{\sqrt{7-6x^3}}}$ (valid if $7 - 6x^3 > 0$, i.e. $x < \left(\frac{7}{6}\right)^{1/3}$).

2. Solve the DE: $\frac{dy}{dx} = \frac{x}{y}$.

S: –Separate variables: $y\,dy = x\,dx$.

–Integrate: $\int y\,dy = \int x\,dx + C$.

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C \quad \Rightarrow y^2 = x^2 + C'.$$

Answer: $\boxed{y = \sqrt{x^2 + C'} \text{ and } y = -\sqrt{x^2 + C'}}$

where C' is an arbitrary constant (valid for x such that $x^2 + C' \geqslant 0$).

---

3. Solve the IVP: $\frac{dy}{dx} = (1 + y^2)\,e^x$ and $y(0) = 0$.

S: –Separate variables: $\frac{dy}{1+y^2} = e^x\,dx$.

–Integrate: $\int \frac{dy}{1+y^2} = \int e^x\,dx + C$.

$$\Rightarrow \tan^{-1} y = e^x + C \quad \Rightarrow y = \tan(e^x + C).$$

–Use the initial condition:

$$0 = \tan(e^0 + C) = \tan(1 + C) \quad \Rightarrow C = -1.$$

Answer: $\boxed{y = \tan(e^x - 1)}$

4. Solve the IVP: $(s+1)\,\frac{ds}{dt} = s\,(1-\sin t)$ and $s(0)=1$.

S: –Separate variables: $\underbrace{\frac{s+1}{s}\,ds}_{\text{only } s} = \underbrace{(1-\sin t)\,dt}_{\text{only } t}.$

–Integrate: $\int \left(1+\frac{1}{s}\right) ds = \int (1-\sin t)\,dt + C.$

$$\Rightarrow\ s + \ln|s| = t + \cos t + C.$$

–Use the initial condition: $s=1$ when $t=0$ so

$$1 + \underbrace{\ln|1|}_{=0} = 0 + \underbrace{\cos 0}_{=1} + C \quad \Rightarrow\ C = 0.$$

Answer: $s + \ln|s| = t + \cos t$

—Note: one cannot get an explicit formula for s in terms of t...

§21.3 Linear First-Order ODEs

$$\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)} \qquad (5)$$

—If (5) is homogeneous, i.e. $Q(x)=0$, then separate variables...

—If (5) is nonhomogeneous, i.e. $Q(x)\neq 0$, see the next Lecture 22...

§22.1 Linear First-Order ODEs

$$\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)} \qquad (*)$$

—Multiply by $v(x)$: $\quad v(x)\,\frac{dy}{dx} + \underbrace{v(x)\,P(x)}_{=\frac{dv}{dx}\text{—choose } v(x) \text{ to satisfy this!}}\,y = v(x)\,Q(x)$

$$\underbrace{v(x)\,\frac{dy}{dx} + v(x)\,P(x)\,y}_{=v\,\frac{dy}{dx}+\frac{dv}{dx}\,y=\frac{d}{dx}(v\,y)}$$

So we observe that:

If $v(x)$ satisfies $\boxed{\frac{dv}{dx} = v\,P(x)}$ (1), then $(*)$ yields $\boxed{\frac{d}{dx}(v\,y) = v\,Q(x)}$ (2).

NOTE: $v(x)$ that satisfies (1) is called an integrating factor for $(*)$.

$\Rightarrow$ To solve $(*)$: —Solve (1) by separating variables:

$$\frac{dv}{dx} = v\,P(x) \;\Rightarrow\; \int \frac{dv}{v} = \int P(x)\,dx \;\Rightarrow\; \ln|v| = \int P(x)\,dx \;\Rightarrow\; \boxed{v = e^{\int P(x)\,dx}}.$$

—Solve (2) (using $v(x)$ already known) as follows:

$$\Rightarrow\; v\,y = \int v(x)\,Q(x)\,dx + C \;\Rightarrow\; \boxed{y(x) = \frac{1}{v(x)}\left(\int v(x)\,Q(x)\,dx + C\right)}$$

§22.2 Examples

1. Solve the IVP: $x\frac{dy}{dx} = x + 2y$ (where $x > 0$), $y(1) = 0$.

S: It is a first-order linear ODE $\Rightarrow$

—Rewrite it in the form $(*)$ as $\frac{dy}{dx} \underbrace{- \frac{2}{x}}_{=P(x)} y = \underbrace{1}_{=Q(x)}$

—Hence the integrating factor is

$$v = e^{\int P(x)\,dx} = e^{\int \left(-\frac{2}{x}\right)\,dx} = e^{-2\ln x} \text{ (where we used } x > 0) \quad \Rightarrow \quad v = \frac{1}{x^2}.$$

—Our DE, multiplied by v, is: $\underbrace{\frac{1}{x^2}\frac{dy}{dx} - \frac{1}{x^2}\frac{2}{x}y}_{=\frac{d}{dx}\left(\frac{1}{x^2}\cdot y\right)} = \frac{1}{x^2}$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{1}{x^2} \quad \Rightarrow \frac{y}{x^2} = \int \frac{1}{x^2} + C = -\frac{1}{x} + C \quad \Rightarrow \boxed{y = -x + C\,x^2}$$

—Use the initial condition:

$y = 0$ at $x = 1$ so $0 = -1 + C \cdot 1^2 \quad \Rightarrow C = 1 \quad \Rightarrow \boxed{y = -x + x^2}$.

2. Solve $\frac{dy}{dx} + \underbrace{x}_{P(x)} y = \underbrace{x^3}_{Q(x)}$.

S: —The integrating factor is $v = e^{\int P(x)\,dx} = e^{\int x\,dx} = e^{x^2/2}$.

—Multiply the DE by v: $e^{x^2/2}\frac{dy}{dx} + \underbrace{x\,e^{x^2/2}}_{(e^{x^2/2})'} y = e^{x^2/2}\,x^3$

$$\Rightarrow \frac{d}{dx}\left(e^{x^2/2}\cdot y\right) = x^3 \cdot e^{x^2/2} \quad \Rightarrow e^{x^2/2}\cdot y = \int \underbrace{x^3 \cdot e^{x^2/2}\,dx}_{\underbrace{x^2}_{u}\cdot\underbrace{x\cdot e^{x^2/2}\,dx}_{dv}}$$

We use $u = x^2 \Rightarrow du = 2x\,dx$, $dv = x\cdot e^{x^2/2}\,dx \Rightarrow v = e^{x^2/2}$, so

$$e^{x^2/2}\cdot y = u\,v - \int v\,du = x^2\cdot e^{x^2/2} - \int e^{x^2/2}\cdot 2x\,dx$$
$$= x^2\cdot e^{x^2/2} - 2\,e^{x^2/2} + C$$

$$\Rightarrow \boxed{y = x^2 - 2 + C\,e^{-x^2/2}}$$ —general solution.

3 Solve $\frac{dy}{dx} + \frac{y}{x} = 1$ (where $x > 0$). S: —The integrating factor is

$$v = e^{\int P(x)\,dx} = e^{\int \frac{dx}{x}} = e^{\ln x} = x \quad \text{(where we used } x > 0\text{)}.$$

—Multiply the DE by $v = x$: $\underbrace{x\frac{dy}{dx} + y}_{=\frac{d}{dx}(x\,y)} = x$

$$\Rightarrow \frac{d}{dx}(x\,y) = x \quad \Rightarrow x\,y = \int x\,dx = \frac{x^2}{2} + C \quad \Rightarrow \boxed{y = \frac{x}{2} + \frac{C}{x}}.$$

4 $(t^2+1)\,\frac{dy}{dt} + t\,y = \frac{1}{2}$. S: —Divide by (t^2+1): $\frac{dy}{dt} + \underbrace{\frac{t}{t^2+1}}_{=P(t)}\,y = \frac{1}{2}\,\frac{1}{t^2+1}$

—The integrating factor is

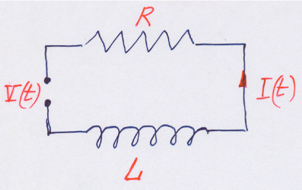
$$v(t) = e^{\int P(t)\,dt} = \underbrace{e^{\int \frac{t}{t^2+1}\,dt}}_{\text{substitute } u=t^2+1} = e^{\frac{1}{2}\ln(t^2+1)} = (t^2+1)^{1/2}.$$

—Multiply the DE by v: $\underbrace{(t^2+1)^{1/2}\,\frac{dy}{dx} + \frac{t}{(t^2+1)^{1/2}}\,y}_{\frac{d}{dt}\left((t^2+1)^{1/2}\cdot y\right)} = \frac{1}{2}\,\frac{1}{(t^2+1)^{1/2}}$

$$\Rightarrow (t^2+1)^{1/2}\cdot y = \frac{1}{2}\int \frac{dt}{(t^2+1)^{1/2}} = \frac{1}{2}\ln\left(t + \sqrt{t^2+1}\right) + C$$

$$\boxed{y = \frac{\ln\left(t+\sqrt{t^2+1}\right)}{2\sqrt{t^2+1}} + \frac{C}{\sqrt{t^2+1}}}$$ —general solution.

§22.3 Example from Electronics: RL-Circuits



The circuit contains a resistor of size R ohms,
an inductor of size L henrys,
a time-varying source of $V(t)$ volts.
$I(t)$ is the current (amperes) at time t.

It is known that the current $I(t)$ satisfies: $\boxed{L\,\frac{dI}{dt} + R\,I = V(t)}$

—a first-order linear DE.

—The integrating factor is $v(t) = e^{\int P(t)\,dt} = e^{\int \frac{R}{L}\,dt} = e^{\frac{Rt}{L}}$.

—Multiply the DE (already divided by L) by v:

$$\underbrace{e^{\frac{Rt}{L}}\,\frac{dI}{dt} + \frac{R}{L}\,e^{\frac{Rt}{L}}\,I}_{\frac{d}{dt}\left(e^{\frac{Rt}{L}}\,I\right)} = e^{\frac{Rt}{L}}\,\frac{V(t)}{L}$$

$$\Rightarrow\ e^{\frac{Rt}{L}}\,I = \int e^{\frac{Rt}{L}}\,\frac{V(t)}{L}\,dt \quad \Rightarrow \boxed{I(t) = \frac{e^{-\frac{Rt}{L}}}{L}\int e^{\frac{Rt}{L}}\,V(t)\,dt}$$

—Now consider a particular case of constant

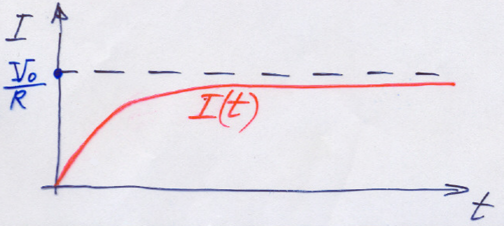
$\boxed{V(t) = V_0}$ and $\boxed{I(0) = 0}$:

i.e. there is a switch in the circuit that is initially open, and then closed at time $t = 0$.

$$\Rightarrow \; I(t) = \frac{e^{-\frac{Rt}{L}}}{L} \int e^{\frac{Rt}{L}} \, V_0 \, dt = \frac{V_0}{L} \, e^{-\frac{Rt}{L}} \left(\frac{L}{R} \, e^{\frac{Rt}{L}} + C \right) = \frac{V_0}{R} \left(1 + C' \, e^{-\frac{Rt}{L}} \right),$$

where $C' = \frac{C\,R}{L}$ is arbitrary.

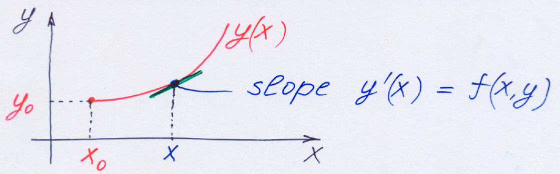
Now use $I(0) = 0 \;\Rightarrow\; 0 = \frac{V_0}{R} \left(1 + C' \, e^{-0} \right) \;\Rightarrow\; C' = -1$



$$\Rightarrow \boxed{I(t) = \frac{V_0}{R} \left(1 - e^{-\frac{Rt}{L}} \right)}$$

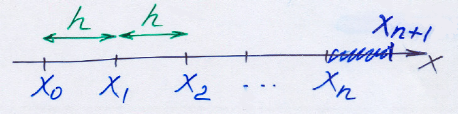
—Note: as $t \to \infty$, one gets $I(t) \to \frac{V_0}{R}$ = the Ohm's Law value!

Consider the Initial-Value Problem $\boxed{y' = f(x, y), \quad y(x_0) = y_0}$ $(*)$



—this can be interpreted as that at each x, the slope of the curve $y(x)$ is $f(x, y)$.

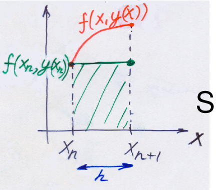
If we can NOT solve $(*)$ explicitly, we can find approximate values $\boxed{y_n \approx y(x_n)}$ at the points $x_n = x_0 + n\,h$:



§23.1 Euler Method

Integrate $y' = f(x, y)$ over the interval $[x_n, x_{n+1}]$:

$$\underbrace{\int_{x_n}^{x_{n+1}} y'\,dx}_{=y(x)\big|_{x_n}^{x_{n+1}} = y(x_{n+1}) - y(x_n)} = \int_{x_n}^{x_{n+1}} f(x, y)\,dx$$



So $y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y(x))\,dx \approx h\,f(x_n, y(x_n))$

—here we used the Rectangular Rule of Numerical Integration

Now, we replace the exact values $y(x_n)$ by approximate values y_n, and also replace $\approx$ by $=$, and so get the definition of a numerical method:

$$\boxed{y(x_{n+1}) \approx y_{n+1} = y_n + h\,f(x_n,\ y_n)} \qquad (**)$$

—called the Euler Method.

NOTE: one can rewrite $(**)$ as

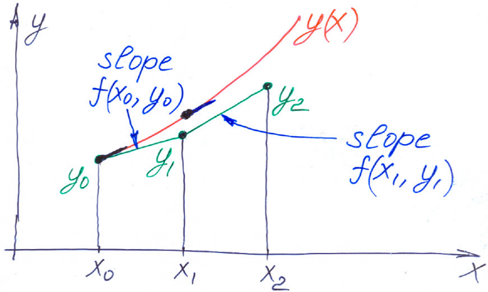
$$\begin{array}{l} y(x_0) = y_0 \text{ (use Initial Condition)} \\ y(x_0 + h) \approx y_1 = y_0 + h\,f(x_0,\ y_0) \\ y(x_0 + 2h) \approx y_2 = y_1 + h\,f(x_1,\ y_1) \\ y(x_0 + 3h) \approx y_3 = y_2 + h\,f(x_2,\ y_2) \cdots \end{array}$$

—i.e. the computation goes as

$$y_0 \mapsto y_1 \mapsto y_2 \mapsto y_3 \mapsto \cdots$$

INTERPRETATION: note that $\frac{y_{n+1}-y_n}{x_{n+1}-x_n} = \underbrace{\frac{y_{n+1}-y_n}{h} = f(x_n,\ y_n)}_{\text{by the Euler method } (**)}$,

i.e. the slope of the computed solution on each (x_n, x_{n+1}) is $f(x_n, y_n)$:



Example: consider the IVP: $y' = x + y$ subject to $y(0) = 0$.

Exercise: show that the exact solution is $y(x) = e^x - x - 1$.

Euler Method: Choose $h = 0.2$ so

$x_n = x_0 + h\,n = 0.2\,n$; $\boxed{y(0) = y_0 = 0}$

Using $f(x, y) = x + y$ one gets

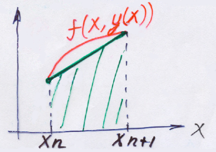
$$\boxed{y(x_{n+1}) = y(0.2[n+1]) \approx y_{n+1} = y_n + h\,f(x_n,\ y_n) = y_n + 0.2\,(x_n + y_n)}$$

n	$x_n = 0.2n$	computed y_n	exact $y(x_n)$	error $y(x_n) - y_n$
0	0	0	0	0
1	0.2	0	0.021	0.021
2	0.4	0.04	0.092	0.052
3	0.6	0.128	0.222	0.094
4	0.8	0.274	0.426	0.152

Error $y(x_n) - y_n$: at each step, the Euler method picks up an error of order h^2, but the error accumulate from step to step, so at $x = x_n$, the error is of order $n \cdot h^2 = \underbrace{(nh)}_{\leqslant \text{length of the interval}} \cdot h = x_n \cdot h \sim h$.

§23.2 The improved Euler Method (Predictor-Corrector)

$$\underbrace{y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y(x))\,dx}_{\text{Recall from §23.1}} \approx \frac{1}{2}\,h\left[f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\right]$$



—here we use the Trapezoidal Rule of Numerical Integration.

If we replace the exact values $y(x_n)$ by approximate values y_n, and also replace $\approx$ by $=$, then we get a numerical method: $y_{n+1} = y_n + \frac{1}{2}\,h\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$.

—This method is expected to be more accurate that the Euler method. However, we have the unknown value y_{n+1} on the right-hand side!!

—To get an easier-to-implement method, replace this y_{n+1} by the Euler approximation from §23.1 (that we now denote y^*_{n+1}), so we arrive at

Improved Euler Method

$y(x_0) = y_0$; $y^*_{n+1} = y_n + h\,f(x_n,\, y_n)$ ="predictor" stage;

$y(x_{n+1}) \approx y_{n+1} = y_n + \frac{1}{2}\,h\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right]$ ="corrector" stage

Example: consider the IVP: $y' = x + y$ subject to $y(0) = 0$.

Improved Euler Method: Choose $h = 0.2$ so $x_n = x_0 + h\,n = 0.2\,n$.

$$y(0) = y_0 = 0$$

Note that $f(x, y) = x + y$ so

$$y^*_{n+1} = y_n + h\,f(x_n,\ y_n) = y_n + 0.2\,(x_n + y_n)$$

Next we get: $$y(x_{n+1}) \approx y_{n+1} = y_n + \tfrac{1}{2}\,h\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right]$$

$$= y_n + 0.1\left[(x_n + y_n) + (x_{n+1} + y^*_{n+1})\right]$$

n	x_n	predictor y^*_n	corrected y_n	exact $y(x_n)$	error $y(x_n) - y_n$
0	0	0	0	0	0
1	0.2	0	0.02	0.0214	0.0014
2	0.4	0.064	0.0884	0.0918	0.0034
3	0.6	0.1861	0.2158	0.2221	0.0063
4	0.8	0.3790	0.4153	0.4255	0.0102

Error $y(x_n) - y_n$: at each step, the Improved Euler method picks up an error of order h^3, but the error accumulate from step to step, so at $x = x_n$, the error is of order $n \cdot h^3 = \underbrace{(nh)}_{\leqslant \text{length of the interval}} \cdot h^2 = x_n \cdot h^2 \sim h^2$.

E.g.: at $x_n = 1$ the error is of order $x_n \cdot h^2 = 1 \cdot h^2 = h^2$ ($x_n = 1$ is a representative choice as it's not too big, not too small).

So the Improved Euler Method is a second-order method (while the Euler method of §23.1 is a first-order method).

In real-world applications, one uses more accurate methods, e.g., Fourth-Order Runge-Kutta methods (see the prime text for further details).

Its error at $x = x_n$ is of order $n \cdot h^5 = \underbrace{(nh)}_{\leqslant \text{length of the interval}} \cdot h^4 = x_n \cdot h^4$;

in particular, at $x_n = 1$, the error is of order h^4.

Another Example: consider the IVP:
$y' = x - y$ subject to $y(0) = 1$ on $[0, 1]$ with the step size $h = 0.2$ using the Euler and the Improved Euler methods.

Exercise: show that the exact solution is $y(x) = x - 1 + 2\, e^{-x}$.

Solution: We have $f(x, y) = x - y$, $x_0 = 0$, and $\boxed{y(0) = y_0 = 1}$ also $h = 0.2$ so $x_n = x_0 + h\, n = 0.2\, n$ for $n = 0, \cdots, 5$.

(a) Euler: $y(x_{n+1}) \approx y_{n+1} = y_n + h\, f(x_n,\, y_n)$,

So $\boxed{y_{n+1} = y_n + 0.2\,(x_n - y_n)}$ subject to $\boxed{y_0 = 1}$.

(b) Improved Euler:

$$\boxed{y^*_{n+1}} = y_n + h\, f(x_n,\, y_n) = \boxed{y_n + 0.2\,(x_n - y_n)} \text{ (as above).}$$

Next we get: $\boxed{y(x_{n+1}) \approx y_{n+1}} = y_n + \frac{1}{2}\, h \left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1}) \right]$

$$\boxed{= y_n + 0.1 \left[(\underbrace{x_n}_{0.2\, n} - y_n) + (\underbrace{x_{n+1}}_{0.2\,(n+1)} - y^*_{n+1}) \right]}$$

Numerical Results:

n	x_n	Euler y_n	Euler error $y(x_n) - y_n$	Improved Euler y_n	Improved Euler error $y(x_n) - y_n$
0	0	1	0	1	0
1	0.2	0.8	0.037	0.84	−0.0025
2	0.4	0.68	0.061	0.7448	−0.0041
3	0.6	0.624	0.073	0.7027	−0.0051
4	0.8	0.619	0.079	0.7042	−0.0055
5	1.0	0.655	0.080	0.7414	−0.0057

the error is over 10% (Euler)   the error is less than 1%! (Improved Euler)

Lecture 24 Second-Order ODEs

(involve second-order derivatives)

—A second-order ODE is called linear if it can be written as

$$\boxed{y'' + p(x)\, y' + q(x)\, y = R(x)} \qquad (*)$$

for some functions $p(x)$ and $q(x)$, called the coefficients.

and some function $R(x)$ called the right-hand side.

Otherwise it is called nonlinear.

E.g., $y'' + x\, y = \cos x$ is linear;
while $y\, y'' + x\, y + y^3 = 0$ is nonlinear.

. . . .

—In the linear case $(*)$:

if $R(x) = 0$, it is called homogeneous;
if $R(x) \neq 0$, it is called nonhomogeneous.

—A general solution of $(*)$ is a formula that describes all solutions of $(*)$ as particular cases.

Example: for the ODE $2x^2\, y'' - x\, y' - 2\, y = 4$, the general solution is

$$y = C_1\, x^2 + C_2\, \frac{1}{\sqrt{x}} - 2 \quad \text{(where } C_1,\ C_2 \text{ are arbitrary constants).}$$

NOTE: A general solution of a linear second-order ODE must involve two arbitrary constants.

. . . .

A particular solution is found by obtaining values for the 2 arbitrary constants from 2 initial or boundary conditions, e.g.,

$$y(x_0) = A, \quad y'(x_0) = B \qquad \text{Initial conditions}$$

$$y(x_0) = A, \quad y(x_1) = B \qquad \text{Boundary conditions}$$

§24.1 Homogeneous Second-Order ODEs

$$\boxed{y'' + p(x)\,y' + q(x)\,y = 0} \qquad (**) \qquad \text{(it's } (*) \text{ with } R(x) = 0\text{)}$$

SUPERPOSITION PRINCIPLE:

If $y_1(x)$ and $y_2(x)$ are any 2 solutions of $(**)$, then $\boxed{A\,y_1(x) + B\,y_2(x)}$ is also a solution of $(**)$ for any real A and B.

Proof:

$$y_1'' + p(x)\,y_1' + q(x)\,y_1 = \mathbf{0}$$

$$y_2'' + p(x)\,y_2' + q(x)\,y_2 = \mathbf{0}$$

$$(A\,y_1 + B\,y_2)'' + p(x)\,(A\,y_1 + B\,y_2)' + q(x)\,(A\,y_1 + B\,y_2) = \mathbf{0}$$

□

A general solution of a homogeneous linear second-order ODE always has the form: $\boxed{y = C_1\, y_1(x) + C_2\, y_2(x)}$

where C_1, C_2 are arbitrary constants,

$y_1(x)$, $y_2(x)$ are two different particular solutions of the ODE

such that $\frac{y_1(x)}{y_2(x)} \neq \text{constant}$.

Such particular solutions are called linearly independent.

CONCLUSION: It suffices to find 2 linearly independent particular solutions to construct a general solution for equation $(**)$.

Example:

for the ODE $2x^2\, y'' - x\, y' - 2\, y = \underline{0}$,

and $y_1(x) = x^2$, $y_2(x) = \frac{1}{\sqrt{x}}$ are two particular solutions,

so a general solution is $y = C_1\, x^2 + C_2\, \frac{1}{\sqrt{x}}$,

where C_1, C_2 are arbitrary constants.

. . . .

§24.2 Linear Second-Order ODEs with Constant Coefficients: Case I Homogeneous ODEs

If the coefficients $p(x)$ and $q(x)$ in $(*)$ are constant, this ODE is called a linear ODE with constant coefficients.

Consider the homogeneous and nonhomogeneous cases separately.

Consider a homogeneous linear second-order ODE with constant coefficients:

$$\boxed{y'' + b\,y' + c\,y = 0} \qquad (**)$$

To find a solution, we make a conjecture that it has the form $\boxed{y = e^{r\,x}}$ with some (unknown at this stage) constant r.

To find r: substitute our guess in $(**)$:

$$y = e^{rx} \quad \Rightarrow \quad y' = r\,e^{r\,x} \quad \Rightarrow \quad y'' = r^2\,e^{r\,x}$$

so substitution in $(**)$ yields: $\quad r^2\,e^{r\,x} + b\,r\,e^{r\,x} + c\,e^{r\,x} = 0 \quad \Rightarrow$

$$\boxed{r^2 + b\,r + c = 0} \qquad (***)$$

—this quadratic equation is called auxiliary (characteristic) for ODE $(**)$.

Its roots are $\boxed{r = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}}$.

Case 1: let $\boxed{b^2 - 4c > 0}$.

Then $(***)$ has 2 distinct real roots r_1 and r_2.

So ODE $(**)$ has 2 different particular solutions $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$.

So general solution is $\boxed{y = C_1\, e^{r_1 x} + C_2\, e^{r_2 x}}$

Case 2: let $\boxed{b^2 - 4c = 0}$. Then $(***)$ has only one root $r_1 = -\frac{b}{2}$
so ODE $(**)$ has a particular solution $y_1 = e^{r_1 x}$.

But to get a general solution, we need another particular solution y_1!

First, rewrite $(**)$ as $\boxed{y'' + b\,y' + \frac{b^2}{4}\,y = 0}$.

Make another solution guess $\boxed{y_2 = x \cdot e^{r_1 x}}$.

To check, whether y_2 is indeed a solution, substitute it into our ODE:

$$\Rightarrow \quad y_2' = (1 + x \cdot r_1)\,e^{r_1 x} \qquad \Rightarrow \quad y_2'' = (2\,r_1 + x \cdot r_1^2)\,e^{r_1 x}$$

Hence

$$y_2'' + b\,y_2' + \frac{b^2}{4}\,y_2 = \underbrace{\left[(2\,r_1 + x \cdot r_1^2) + b\,(1 + x \cdot r_1) + \frac{b^2}{4}\,x\right]}_{=0 \text{ as } r_1 = -\frac{b}{2} \text{ (Ex.)}}\,e^{r_1 x} = 0$$

Hence $y_2(x)$ is indeed another particular solution of $(**)$ so the general solution $y = C_1\,y_1(x) + C_2\,y_2(x)$ becomes $\boxed{y = (C_1 + C_2\,x)\,e^{r_1 x}}$.

Case 3: let $\boxed{b^2 - 4c < 0}$

—similar to Case 1: $(***)$ has 2 distinct roots,

but they are complex: $r_1 = \alpha + i\,\beta$ and $r_2 = \alpha - i\,\beta$.

As in Case 1, the ODE $(**)$ has a general solution

$$y = C_1\, e^{r_1 x} + C_2\, e^{r_2 x} = C_1\, e^{(\alpha + i\,\beta)\,x} + C_2\, e^{(\alpha - i\,\beta)\,x}$$

where C_1, C_2 are arbitrary complex constants.

To restrict this formula to real functions and constants, recall that

$$e^{(\alpha + i\,\beta)\,x} = e^{\alpha\,x} e^{i\,\beta\,x} = e^{\alpha\,x}\big(\cos(\beta\,x) + i\,\sin(\beta\,x)\big)$$

$$e^{(\alpha - i\,\beta)\,x} = e^{\alpha\,x} e^{-i\,\beta\,x} = e^{\alpha\,x}\big(\cos(\beta\,x) - i\,\sin(\beta\,x)\big)$$

These imply: $y(x) = e^{\alpha\,x}\Big(\underbrace{(C_1 + C_2)}_{C_1'}\cos(\beta\,x) + \underbrace{i\,(C_1 - C_2)}_{C_2'}\sin(\beta\,x)\Big)$

Finally, a general solution of $(**)$ is written as

$$\boxed{y = e^{\alpha\,x}\big(C_1'\,\cos(\beta\,x) + C_2'\,\sin(\beta\,x)\big)}$$

(where C_1', C_2' are arbitrary real constants).

For a homogeneous linear second-order ODE with constant coefficients: $y'' + b\,y' + c\,y = 0$ $(**)$

Summary:	Roots of $(***)$	General Solution of $(**)$
(1) $b^2 - 4c > 0$	2 real roots: $r_1,\ r_2$	$y = C_1\, e^{r_1 x} + C_2\, e^{r_2 x}$
(2) $b^2 - 4c = 0$	1 real root $r_1 = -\frac{b}{2}$	$y = (C_1 + C_2\, x)\, e^{r_1 x}$
(3) $b^2 - 4c < 0$	2 complex roots: $r_{1,2} = \alpha \pm i\,\beta$	$y = e^{\alpha x}\big(C_1 \cos(\beta\, x) + C_2 \sin(\beta\, x)\big)$

where $r^2 + b\,r + c = 0$ $(***)$

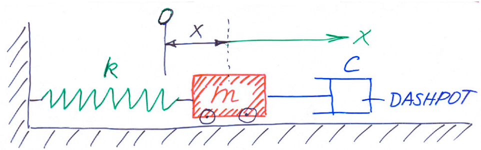
Examples:

1. $y'' + y' - 2y = 0$. S: $r^2 + r - 2 = 0 \quad \Rightarrow \ r = \frac{-1 \pm \sqrt{1^2 - 4\cdot(-2)}}{2}$
   $r_1 = -2,\ r_2 = 1 \Rightarrow\ y_1 = e^{-2x},\ y_2 = e^{x}$ $\boxed{y = C_1\, e^{-2x} + C_2\, e^{x}}$

2. $16\,y'' - 8\,y' + y = 0$. S: $16\,r^2 - 8\,r + 1 = 0$
   $\Rightarrow\ r = \frac{8 \pm \sqrt{8^2 - 4\cdot 16}}{2\cdot 16} = \frac{1}{4}$ —single real root $\boxed{y = (C_1 + C_2\, x)\, e^{x/4}}$

3. $y'' + 4\,y' + 13\,y = 0$
   S: $r^2 + 4\,r + 13 = 0 \quad \Rightarrow\ r = \frac{-4 \pm \sqrt{4^2 - 4\cdot 13}}{2} = -2 \pm \sqrt{-9}$
   $r = -2 \pm 3\,i$: $r_1 = -2 + 3\,i$, $r_2 = -2 - 3\,i$ (i.e. $\alpha = -2$, $\beta = 3$)
   $\boxed{y = e^{-2\,x}\left(C_1\, \cos(3\,x) + C_2\, \sin(3\,x)\right)}$

§24.3 Physical Example: Damped Mass-Spring System



—A mass m is attached to a spring with spring constant k, and to a dashpod with damping coefficient c.
Its displacement at time t (w.r.t. the rest point) is $x(t)$.

RECALL: By Newton's Second Law: mass × acceleration = force

$$m \times a = \underbrace{-k \cdot x}_{\text{Hook'sforce is proportional to } x} \quad \underbrace{- c \cdot v}_{\text{damping force is proportional to } v}$$

—Here $a = \frac{d^2x}{dt^2}$ is the acceleration, $v = \frac{dx}{dt}$ is the velocity.

So $x(t)$ satisfies the ODE: $\boxed{m\,\frac{d^2x}{dt^2} + c\,\frac{dx}{dt} + k\,x = 0}$

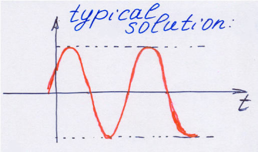
(with positive constants $m,\ c,\ k > 0$).

(a) $\boxed{c = 0}$, i.e. no damping:

called simple harmonic motion,

$m\,r^2 + 0 \cdot r + k = 0$ has complex roots $r_{1,2} = \pm i\sqrt{\frac{k}{m}}$

$\Rightarrow\ x(t) = C_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right)$



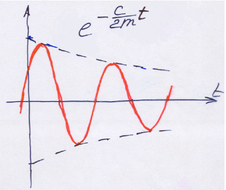
(b) $\boxed{c > 0,\ c^2 < 4mk}$, i.e. Damped Oscillator:

$$\boxed{m\,r^2 + c\,r + k = 0} \quad (****)$$

has roots $\boxed{r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}}$

i.e. $r_{1,2} = -\frac{c}{2m} \pm i\,\beta$ with $\beta = \frac{\sqrt{4mk - c^2}}{2m}$

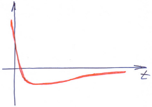
$\Rightarrow\ x(t) = e^{-\frac{c}{2m}t}\left(C_1 \cos(\beta\,t) + C_2 \sin(\beta\,t)\right)$



(c) $\boxed{c^2 = 4mk}$, i.e. Critically Damped case:

$(****)$ has a single root: $r_1 = -\frac{c}{2m}$

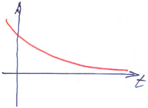
$\Rightarrow\ x(t) = (C_1 + C_2\,t)\,e^{-\frac{c}{2m}t}$ —no oscillations



(d) $\boxed{c^2 > 4mk}$, i.e. Overdamped case:

$(****)$ has 2 real roots $r_1 < r_2 < 0$

$\Rightarrow\ x(t) = C_1\,e^{r_1 t} + C_2\,e^{r_2 t}$ —decay, no oscillations...



Lecture 25 Linear Second-Order ODEs with Constant Coefficients: Case II Nonhomogeneous ODEs

Consider a nonhomogeneous linear second-order ODE

$$\boxed{y'' + p(x)\,y' + q(x)\,y = \mathbf{R(x)}} \qquad (*)$$

The corresponding homogeneous equation: $\boxed{y_h'' + p\,y_h' + q\,y_h = \mathbf{0}}$ $\quad (**)$

—If $y_p(x)$ is any particular solution of $(*)$, while $y_h(x)$ is any solution of $(**)$, then $\boxed{y_p(x) + y_h(x)}$ is also a solution of $(*)$.

Proof:

$$\begin{array}{rr} & y_p'' + p\,y_p' + q\,y_p = \mathbf{R(x)} \\ & y_h'' + p\,y_h' + q\,y_h = \;\mathbf{0} \\ \hline \Rightarrow & (y_p + y_h)'' + p\,(y_p + y_h)' + q\,(y_p + y_h) = \mathbf{R(x)} \end{array} \qquad \square$$

Hence, general solution of nonhomogeneous equation $(*)$ = particular solution of $(*)$ + general solution of homogeneous equation $(**)$

(this must involve 2 arbitrary constants)

$\Rightarrow$ To find a general solution of nonhomogeneous equation $(*)$:

Step 1: Find a general solution of the corresponding homogeneous equation $(**)$ (e.g., as in §24.2).

Step 2: Add a particular solution of $(*)$.

—To find this, there are various methods; one of them is

Method of Undetermined Coefficients (see below)

Method of Undetermined Coefficients

— applies to the constant-coefficient $\boxed{y'' + b\,y' + c\,y = \mathbf{R}(\mathbf{x})}$ $\quad(*)$

$\mathbf{R}(\mathbf{x})$	Try $y_p(x) =$
α	A
$\alpha\,x + \beta$	$A\,x + B$
$\alpha\,x^2 + \beta\,x + \gamma$	$A\,x^2 + B\,x + C$
polynomial of degree n	another poly of the same degree n
$\alpha\,e^{kx}$	$A\,e^{kx}$
$(\alpha + \beta\,x)\,e^{kx}$	$(A\,x + B)\,e^{kx}$
$\alpha\,\cos(kx) + \beta\,\sin(kx)$	$A\,\cos(kx) + B\,\sin(kx)$
$(\alpha\,x + \beta)\cos(kx) + (\gamma\,x + \delta)\sin(kx)$	$(A\,x + B)\cos(kx) + (C\,x + D)\sin(kx)$

Make this guess, substitute in $(*)$,

then choose A, B, C ... in $y_p(x)$ so that $y_p'' + b\,y_p' + c\,y_p = \mathbf{R}(\mathbf{x})$...

Examples:

1. $y'' + y = x^2$

Step 1: $y_h'' + y_h = 0$ —homogeneous

$r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm i \quad \Rightarrow \quad \boxed{y_h = C_1 \cdot \cos x + C_2 \cdot \sin x}$

Step 2: We guess that $y_p = A\,x^2 + B\,x + C$ (see the Table)

$$\Rightarrow \quad y_p' = 2A\,x + B \quad \Rightarrow \quad y_p'' = 2A$$

Substitute in the ODE: $\underbrace{2A}_{=y_p''} + \underbrace{(A\,x^2 + B\,x + C)}_{=y_p} = x^2$

$$\Rightarrow \quad A\,x^2 + B\,x + (2A + C) = 1 \cdot x^2$$

$$\Rightarrow \quad A = 1, \quad B = 0, \quad (2A + C) = 0 \quad \Rightarrow \quad C = -2A = -2$$

$$\Rightarrow \quad y_p = 1 \cdot x^2 - 2 + 0 \cdot x = \boxed{x^2 - 2}$$

Finally, a general solution:

$$\boxed{y = (x^2 - 2) + C_1 \cdot \cos x + C_2 \cdot \sin x}$$

2. $y'' - y' - 2y = 10\cos t$ subject to $y(0) = 0$ and $y'(0) = 2$.

Plan: (i) find a general solution of the ODE; (ii) use the initial conditions.

(i) Step 1: $y_h'' - y_h' - 2y_h = 0$ —homogeneous

$$r^2 - r - 2 = 0 \quad r_1 = 2, \quad r_2 = -1 \;\Rightarrow\; \boxed{y_h = C_1 e^{2t} + C_2 e^{-t}}$$

Step 2: Our guess is $y_p = A\cos t + B\sin t$

$$\Rightarrow y_p' = -A\sin t + B\cos t \quad \Rightarrow y_p'' = -A\cos t - B\sin t$$

$$\text{So } y_p'' - y_p' - 2y_p = \cos t\big(-A - B - 2A\big) + \sin t\big(-B + A - 2B\big)$$

$$= \cos t\big(-3A - B\big) + \sin t\big(A - 3B\big) = 10\cos t$$

$$\left.\begin{aligned} -3A - B &= 10 \\ A - 3B &= 0 \end{aligned}\right\} \Rightarrow B = -1, \quad A = -3 \quad \Rightarrow \boxed{y_p = -3\cos t - \sin t}$$

$$\Rightarrow \text{General solution: } \boxed{y = (-3\cos t - \sin t) + C_1 e^{2t} + C_2 e^{-t}}$$

Example 2 (continued): **(ii)** Use initial conditions

$y(0) = 0$: $\Rightarrow\ 0 = (-3\underbrace{\cos 0}_{=1} - \underbrace{\sin 0}_{=0}) + C_1 \cdot 1 + C_2 \cdot 1 \ \Rightarrow\ C_1 + C_2 = 3$

$y'(0) = 2$: Note that $y' = (3\sin t - \cos t) + 2\,C_1\,e^{2t} - C_2\,e^{-t}\ \Rightarrow$

$$2 = (3\underbrace{\sin 0}_{=0} - \underbrace{\cos 0}_{=1}) + 2\,C_1\,1 - C_2\,1 \ \Rightarrow\ 2\,C_1 - C_2 = 3$$

$$\left.\begin{array}{r} C_1 + C_2 = 3 \\ 2\,C_1 - C_2 = 3 \end{array}\right\} \Rightarrow\ C_1 = 2,\ C_2 = 1 \Rightarrow \boxed{y = (-3\cos t - \sin t) + 2\,e^{2t} + e^{-t}}$$

Remark 1: Consider $\boxed{y'' + b\,y' + c\,y = \mathbf{R_1}(\mathbf{x}) + \mathbf{R_2}(\mathbf{x})}$ $(***)$

If $y_1'' + b\,y_1' + c\,y_1 = \mathbf{R_1}(\mathbf{x})$ and $y_2'' + b\,y_2' + c\,y_2 = \mathbf{R_2}(\mathbf{x})$

then $y_1 + y_2$ is a particular solution of $(***)$.

Further Examples:

3. $y'' + 4y = \sin t + 2\,e^{-t}$. Step 1: $y_h'' + 4y_h = 0$—homogeneous

$$r^2 + 4 = 0, \qquad r = \pm 2\,i \quad \Rightarrow \quad \boxed{y_h = C_1 \cos(2t) + C_2 \sin(2t)}$$

Step 2:

	$\mathbf{R(x)}$	Try $y_p(x) =$
Our Table $\Rightarrow$	$\sin t$ e^{-t}	$A\cos t + B\sin t$ $C\,e^{-t}$
Remark 1 $\Rightarrow$	$\sin t + 2\,e^{-t}$	$A\cos t + B\sin t + C\,e^{-t}$

So $y_p = A\cos t + B\sin t + C\,e^{-t} \quad \Rightarrow \quad y_p'' = -A\cos t - B\sin t + C\,e^{-t}$

$$y_p'' + 4y_p = \left(-A\cos t - B\sin t + C\,e^{-t}\right) + 4\left(A\cos t + B\sin t + C\,e^{-t}\right)$$

$$= 3\,A\cos t + 3\,B\sin t + 5\,C\,e^{-t} = \sin t + 2\,e^{-t}$$

$$\left.\begin{array}{l} 3\,A = 0 \\ 3\,B = 1 \\ 5\,C = 2 \end{array}\right\} \Rightarrow \; A = 0,\; B = \tfrac{1}{3},\; C = \tfrac{2}{5} \;\Rightarrow\; \boxed{y_p = \tfrac{1}{3}\sin t + \tfrac{2}{5}\,e^{-t}}$$

General solution: $\boxed{y = \tfrac{1}{3}\sin t + \tfrac{2}{5}\,e^{-t} + C_1\cos(2t) + C_2\sin(2t)}$

§25.3 (Very Important) Remark 2

—If the guess from the Table happens to be a particular solution of the homogeneous equation $(**)$, then use $\boxed{y_p = x \cdot (\text{guess from Table})}$.

—If the new guess also happens to be a particular solution of the homogeneous equation $(**)$, then use $\boxed{y_p = x^2 \cdot (\text{guess from Table})}$

Further Examples:

4. $y'' - 2\,y' - 3\,y = e^{-x}$

Step 1: $\boxed{y_h = C_1\,e^{3x} + C_2\,e^{-x}}$ (check!)

Step 2: The Table suggests: $\boxed{y_p = A\,e^{-x}}$, but it is a particular case of y_h:

—it's clear from the y_h formula (set $C_1 = 0$ and $C_2 = A$);

—alternatively, one can see this directly:

$$(A\,e^{-x})'' - 2\,(A\,e^{-x})' - 3\,(A\,e^{-x}) = 0 \neq e^{-x}.$$

$\Rightarrow$ Clearly, $y_p = A\,e^{-x}$ doesn't work!

By Remark 2, try $\boxed{y_p = x \cdot A\,e^{-x}}$:

$$y_p' = A\,(1-x)\,e^{-x}, \quad y_p'' = A\,(x-2)\,e^{-x}$$

$$y_p'' - 2\,y_p' - 3\,y_p = A\Big((x-2) - 2\,(1-x) - 3\,x\Big)\,e^{-x} = A\,(-4)\,e^{-x} = e^{-x}$$

$$-4A = 1 \Rightarrow A = -\tfrac{1}{4} \Rightarrow \boxed{y_p = -\tfrac{1}{4}\,x\,e^{-x}} \Rightarrow \boxed{y = -\tfrac{1}{4}\,x\,e^{-x} + C_1\,e^{3x} + C_2\,e^{-x}}$$

5. $y'' + 4y' + 4y = e^{-2t}$

Step 1: $r^2 + 4r + 4 = 0 \qquad r = -2$—single root

$$\Rightarrow \boxed{y_h = (C_1 + C_2\, t)\, e^{-2t}}$$

Step 2: The Table suggests $\boxed{y_p = A\, e^{-2t}}$

But $(A\, e^{-2t})'' + 4(A\, e^{-2t})' + 4(A\, e^{-2t}) = 0, \ \neq e^{-2t}$

(can be checked directly; of from the y_h formula...)

$\Rightarrow$ By Remark 2, try $\boxed{y_p = t \cdot A\, e^{-2t}}$ —again won't work as

$(t \cdot A\, e^{-2t})'' + 4(t \cdot A\, e^{-2t})' + 4(t \cdot A\, e^{-2t}) = 0, \ \neq e^{-2t}$

(can be checked directly; of from the y_h formula...)

$\Rightarrow$ By Remark 2, now try $\boxed{y_p = t^2 \cdot A\, e^{-2t}}$

$$y_p' = (2t - 2t^2) \cdot A\, e^{-2t}, \qquad y_p'' = (2 - 8t + 4t^2) \cdot A\, e^{-2t}$$

$$y_p'' + 4y_p' + 4y_p = \big((2 - 8t + 4t^2) + 4\,(2t - 2t^2) + 4\, t^2\big) \cdot A\, e^{-2t}$$

$$= 2\, A\, e^{-2t} = e^{-2t}$$

$$\Rightarrow A = \tfrac{1}{2} \Rightarrow \boxed{y_p = \tfrac{1}{2}\, t^2\, e^{-2t}} \Rightarrow \boxed{y = \tfrac{1}{2}\, t^2\, e^{-2t} + (C_1 + C_2\, t)\, e^{-2t}}$$

6. $y'' + 4y = 8\cos(2t)$

Step 1: $\boxed{y_h = C_1\cos(2t) + C_2\sin(2t)}$ (see Example 3).

Step 2: The Table suggests $\boxed{y_p = A\cos(2t) + B\sin(2t)}$

But will NOT work as it's a particular case of the y_h formula...

$\Rightarrow$ By Remark 2, try $\boxed{y_p = t\cdot\big(A\cos(2t) + B\sin(2t)\big)}$

$$\Rightarrow\ y_p'' + 4y_p = 4B\cos(2t) + 4(-A)\sin(2t) = 8\cos(2t)$$

$$\Rightarrow\ B = 2,\quad A = 0\quad \text{so}\quad \boxed{y_p = 2t\sin(2t)}$$

Finally, we get the

Answer: $\boxed{y = 2t\sin(2t) + C_1\cos(2t) + C_2\sin(2t)}$

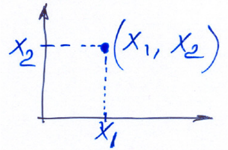
NOTE: this phenomenon is referred to as resonance

(see the prime text....)

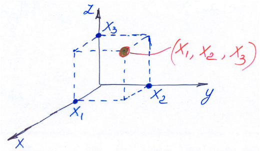
$$\boxed{\mathbb{R}^n = \{(x_1, x_2, x_3, \cdots, x_n) \text{ where each } x_i \in \mathbb{R},\ i = 1, 2, \cdots, n\}}$$

—is the set of ordered *n*-tuplets of real numbers;
this space is called Euclidean *n*-space.

- $\mathbb{R}^2$ (here $n = 2$):
  each (x_1, x_2) is represented
  by a point on the plane:



- $\mathbb{R}^3$ (here $n = 3$):
  each (x_1, x_2, x_3) is represented
  by a point in space:



- $\mathbb{R}^4$ (here $n = 4$): each point (x_1, x_2, x_3, x_4) is an algebraic object
  (i.e. no obvious geometric representation).

  E.g.: $(1, -1, 0, \frac{1}{2})$ and $(8, 1.2, -5, 1)$ are both elements of $\mathbb{R}^4$.

Definition

A function f of n real variables is a rule
that assigns a unique real number, denoted $f(x_1, x_2, x_3, \cdots, x_n)$
to each point $(x_1, x_2, x_3, \cdots, x_n)$ in the n-space $\mathbb{R}^n$.

Example:

a function $f : \mathbb{R}^3 \mapsto \mathbb{R}$ is defined by $f(x_1, x_2, x_3) = x_1^2 - 2x_2x_3$.

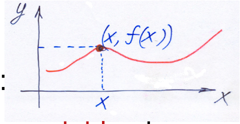
Now, e.g., $f(1, 2, 3) = 1^2 - 2 \cdot 2 \cdot 3 = -11$.

....

NOTE: For simplicity, we mainly consider functions of two variables, so we denote these variables by x and y (rather than x_1 and x_2), while the values of the function are denoted by z so $\boxed{z = f(x, y)}$.

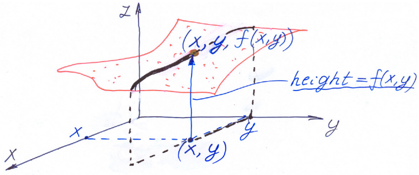
Graphical Representation:

—A function $y = f(x)$ of one variable
is represented by a curve on the plane:



—Similarly, a function $z = f(x, y)$ of two variables is represented by surface in space, obtained as follows:

for each pair $(x, y) \in \mathbb{R}^2$, use $z = f(x, y)$
as the "signed" height above/below
the (x, y)-plane:



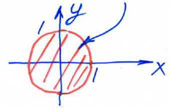
. . . .

—Definitions of limits, continuity, ... can be extended
to functions of several variables.

—Similarly, $f(x, y)$ may be defined for all $(x, y) \in \mathbb{R}^2$,
or on some subset $D \subset \mathbb{R}^2$ called the domain of f.

E.g.: the function $f(x, y) = x + \sqrt{1 - (x^2 + y^2)}$ has domain:

$D = \{(x, y) \text{ such that } x^2 + y^2 \leqslant 1\}$ —unit disk.



§26.2 Partial Differentiation

The partial derivative of $f(x,y)$ w.r.t. x at (x_0, y_0) is

$$\boxed{\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}}$$ (provided this limit exists).

Interpretation: we freeze $y = y_0$ and differentiate the function $f(x, y_0)$ of one variable x in the standard way w.r.t. x.

Thus $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ is the rate of change (slope) of f as we move from (x_0, y_0) in the x-direction.

The partial derivative of $f(x,y)$ w.r.t. y at (x_0, y_0) is

$$\boxed{\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{k\to 0}\frac{f(x_0,y_0+k)-f(x_0,y_0)}{k}}$$ (provided this limit exists).

Interpretation: we freeze $x = x_0$ and differentiate the function $f(x_0, y)$ of one variable y in the standard way w.r.t. y.

Thus $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$ is the rate of change (slope) of f as we move from (x_0, y_0) in the y-direction.

Notation: for the function $z = f(x,y)$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}f(x,y) = f_x(x,y)$$
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}f(x,y) = f_y(x,y)$$

To evaluate $\frac{\partial f}{\partial x}$: differentiate w.r.t. x treating y as a constant.

E.g.: $\frac{d}{dx}(e^{2x} - 3x^2) = 2\,e^{2x} - 3 \cdot 2\,x.$

Similarly, $\frac{\partial}{\partial x}(e^{yx} - y^2\,x^2) = y\,e^{yx} - y^2 \cdot 2\,x.$

To evaluate $\frac{\partial f}{\partial y}$: differentiate w.r.t. y treating x as a constant.

E.g.: $\frac{d}{dy}\sin(k\,y^2) = \cos(k\,y^2) \cdot \frac{d}{dy}(k\,y^2) = \cos(k\,y^2) \cdot k \cdot 2y.$

Similarly, $\frac{\partial}{\partial y}\sin(x^3\,y^2) = \cos(x^3\,y^2) \cdot \frac{\partial}{\partial y}(x^3\,y^2) = \cos(x^3\,y^2) \cdot x^3 \cdot 2y.$

Examples:

1. $f(x,y) = 2x + 3y + x^2 y + e^x \sin y$
   $\frac{\partial f}{\partial x} = 2 + 0 + 2xy + e^x \sin y$; $\frac{\partial f}{\partial y} = 0 + 3 + x^2 + e^x \cos y.$

2. The pressure in an ideal gas is $p = \frac{T}{V}$. Find the rate of change of pressure p: (i) with temperature T; (ii) with volume V, when $T = 100$ and $V = 1$.

S: (i) $\frac{\partial p}{\partial T} = \frac{1}{V}$; $\left.\frac{\partial p}{\partial T}\right|_{(100,1)} = \frac{1}{1} = 1$ (units of pressure per unit temperature).

(ii) $\frac{\partial p}{\partial V} = -\frac{T}{V^2}$; $\left.\frac{\partial p}{\partial V}\right|_{(100,1)} = -\frac{100}{1^2} = -100$ (units of pressure per unit volume). □

§26.3 Second Partial Derivatives

Assuming the limits exist, define:

- Pure second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} \quad \text{—w.r.t. } x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy} \quad \text{—w.r.t. } y$$

- Mixed second partial derivatives

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx} \quad \text{—this is } (f_y)_x$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} \quad \text{—this is } (f_x)_y$$

NOTE: here ”interior” differentiation occurs first!!!

Example: Find all first and second partial derivatives for

$$f(x,y) = 3x^2 - 2xy^2 + \sin(xy).$$

S: $\frac{\partial f}{\partial x} = 6x - 2y^2 + y\cos(xy)$; $\frac{\partial f}{\partial y} = 0 - 4xy + x\cos(xy)$;

$$\frac{\partial^2 f}{\partial x^2} = 6 - 0 + y\left(-y\sin(xy)\right) = 6 - y^2\sin(xy);$$

$$\frac{\partial^2 f}{\partial y^2} = -4x + x\left(-x\sin(xy)\right) = -4x - x^2\sin(xy);$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(-4xy + x\cos(xy)\right) = -4y + \underbrace{\left[\cos(xy) - x\,y\sin(xy)\right]}_{\text{Product Rule}};$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(6x - 2y^2 + y\cos(xy)\right) = 0 - 4y + \underbrace{\left[\cos(xy) - x\,y\sin(xy)\right]}_{\text{Product Rule}}.$$

□

NOTE: in this example $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$ —it is NOT a coincidence! ⇓

The Mixed Derivative Theorem

If $f(x,y)$ and all its first and second partial derivatives are defined and continuous in a region R, then in this region:

$$\boxed{\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}} \quad \text{or, equivalently,} \quad \boxed{f_{yx} = f_{xy}}.$$

§27.1 Taylor Series in Two Variables

Recall: for one variable we have the Taylor series expansion:

$$f(x+h) = f(x) + h\,f'(x) + \frac{1}{2!}\,h^2\,f''(x) + \cdots$$

(this formula is useful only when h is small).

The generalization to two variables:

$$f(x+h, y+k) = f(x,y) + \left(h\,\frac{\partial f}{\partial x}(x,y) + k\,\frac{\partial f}{\partial y}(x,y)\right) + \frac{1}{2!}\left(h^2\,\frac{\partial^2 f}{\partial^2 x}(x,y) + 2\,h\,k\,\frac{\partial^2 f}{\partial x\,\partial y}(x,y) + k^2\,\frac{\partial^2 f}{\partial y^2}(x,y)\right) + \cdots$$

where $\cdots$ stands for higher-order terms;

(this formula is useful only when both h and k are small).

NOTE: we can write this formula in a more compact way:

$$f(x+h, y+k) = f(x,y) + \left(h\,f_x(x,y) + k\,f_y(x,y)\right) + \frac{1}{2!}\left(h^2\,f_{xx}(x,y) + 2\,h\,k\,f_{xy}(x,y) + k^2\,f_{yy}(x,y)\right) + \cdots$$

Examples: Find the Taylor series up to quadratic terms for the given functions.

1. $f(x,y) = x^2 \sin y$ about the point $(1,0)$.

S:

$f = x^2 \sin y$	$f(1,0) = 1^2 \sin 0 = 0$
$f_x = 2x \sin y$ $f_y = x^2 \cos y$	$f_x(1,0) = 0$ $f_y(1,0) = 1$
$f_{xx} = 2 \sin y$ $f_{xy} = 2x \cos y$ $f_{yy} = -x^2 \sin y$	$f_{xx}(1,0) = 0$ $f_{xy}(1,0) = 2$ $f_{yy}(1,0) = 0$

$$f(1+h, 0+k) = \underbrace{f(1,0)}_{=0} + \Big(h \underbrace{f_x(1,0)}_{=0} + k \underbrace{f_y(1,0)}_{=1}\Big)$$

$$+ \frac{1}{2!}\Big(h^2 \underbrace{f_{xx}(1,0)}_{=0} + 2hk \underbrace{f_{xy}(1,0)}_{=2} + k^2 \underbrace{f_{yy}(1,0)}_{=0}\Big) + \cdots$$

$$= 0 + h \cdot 0 + k \cdot 1 + \frac{1}{2}\left(h^2 \cdot 0 + 2hk \cdot 2 + k^2 \cdot 0\right) + \cdots$$

Answer: $\boxed{f(1+h,k) = k + 2hk + \cdots}$

2. $f(x,y)=\sqrt{x^2+y^3}$ about the point $(1,2)$.

S:

$$f(1,2)=\sqrt{1^2+2^3}=3$$

$$f_x=\frac{x}{\sqrt{x^2+y^3}} \Rightarrow f_x(1,2)=\frac{1}{3}$$

$$f_y=\frac{3y^2}{2\sqrt{x^2+y^3}} \Rightarrow f_y(1,2)=\frac{3\cdot 2^2}{2\cdot 3}=2$$

$$f_{xx}=\frac{y^3}{(x^2+y^3)^{3/2}} \Rightarrow f_{xx}(1,2)=\frac{8}{27}$$

$$f_{xy}=\frac{-3xy^2}{2\,(x^2+y^3)^{3/2}} \Rightarrow f_{xy}(1,2)=-\frac{2}{9}$$

$$f_{yy}=\frac{12x^2y+3y^4}{4\,(x^2+y^3)^{3/2}} \Rightarrow f_{yy}(1,2)=\frac{2}{3}$$

—exercise!

$$f(1+h,2+k)=\underbrace{f(1,2)}_{=3}+\Big(h\,\underbrace{f_x(1,2)}_{=\frac{1}{3}}+k\,\underbrace{f_y(1,2)}_{=2}\Big)$$

$$+\frac{1}{2!}\Big(h^2\,\underbrace{f_{xx}(1,2)}_{=\frac{8}{27}}+2\,h\,k\,\underbrace{f_{xy}(1,2)}_{=-\frac{2}{9}}+k^2\,\underbrace{f_{yy}(1,2)}_{=\frac{2}{3}}\Big)+\cdots$$

$$=\boxed{3+h\cdot\frac{1}{3}+k\cdot 2+\frac{1}{2}\left(h^2\cdot\frac{8}{27}-\frac{4}{9}\,h\,k+k^2\cdot\frac{2}{3}\right)+\cdots}.$$

How to use this formula?? —dropping higher-order terms $\cdots$, we get

$$f(1+h, 2+k) \approx 3 + \frac{1}{3}h + 2k + \frac{1}{2}\left(\frac{8}{27}h^2 - \frac{4}{9}hk + \frac{2}{3}k^2\right)$$

for small h, k.

E.g.: To get $f(1.02, 1.97) = f(1 + 0.02, 2 - 0.03)$,

so use $h = 0.02$ and $k = -0.03$, which yields $f(1.02, 1.97) \approx \underbrace{2.94715}_{\text{correct}}9$

—very accurate (6 correct decimal places!)

NOTE:

Taylor series give approximations of a function near a certain point:

—The closer the point at which we evaluate f to the point about which the Taylor series is constructed, the more accurate is the approximation.

—The more terms are used, the more accurate is the approximation (although more complicated).

—Sometimes, fewer terms in the Taylor series may give a sufficiently accurate approximation: see the next §27.2...

§27.2 Linear Aproximation

To approximate f near a certain point (x, y), we now drop the second-order and higher-order terms in its Taylor series (so take fewer terms than in §27.1):

$$\boxed{f(x+h, y+k) \approx \underbrace{f(x,y) + h\, f_x(x,y) + k\, f_y(x,y)}}$$

called linear approximation of f at (x, y) (or linearization)

Examples:

1. Using the linearization of $f(x,y) = \sqrt{2x^2 + e^{2y}}$ at $(2,0)$, find an approximate value of $f(2.2, -0.2)$.

S: **(i)** Construct the linearization: $f(2,0) = 3$,

$$f_x = \frac{2x}{\sqrt{2x^2+e^{2y}}} \Rightarrow f_x(2,0) = \frac{4}{3}, \qquad f_y = \frac{e^{2y}}{\sqrt{2x^2+e^{2y}}} \Rightarrow f_y(2,0) = \frac{1}{3},$$

$$f(2+h, 0-k) \approx f(2,0) + h\, f_x(2,0) + k\, f_y(2,0) = 3 + \frac{4}{3}\, h + \frac{1}{3}\, k.$$

(ii) $$f(2.2, -0.2) = f(2 + \underbrace{0.2}_{h}, 0 \underbrace{-\, 0.2}_{k}) \approx 3 + \frac{4}{3} \cdot 0.2 + \frac{1}{3} \cdot (-0.2) = 3.2.$$

Remark 1: Change the notation to $h = \triangle x$, $k = \triangle y$:

$$\underbrace{f(x + \triangle x, y + \triangle y) - f(x, y)}_{=\triangle z} \approx \triangle x \cdot f_x(x, y) + \triangle y \cdot f_y(x, y)$$

Here we also used the notation: $\boxed{\triangle z = f(x + \triangle x, y + \triangle y) - f(x, y)}$

—very natural notation for change in $z = f(x, y)$
that corresponds to change $\triangle x$ in x and change $\triangle y$ in y.

Then we get an alternative representation
of our Linear Approximation formula:

$$\boxed{\triangle z \approx f_x(x, y)\,\triangle x + f_y(x, y)\,\triangle y = \frac{\partial f}{\partial x}\,\triangle x + \frac{\partial f}{\partial y}\,\triangle y}$$

Remark 2:

Similarly, for a function of 3 variables $w = f(x, y, z)$ we have

$$\boxed{\triangle w \approx f_x(x, y, z)\,\triangle x + f_y(x, y, z)\,\triangle y + f_z(x, y, z)\,\triangle z}$$

where $\boxed{\triangle w = f(x + \triangle x, y + \triangle y, z + \triangle z) - f(x, y, z)}$

Examples:

2. If $w = \frac{x^2 y^4}{z^3}$, find the approximate change in w if: x increases by 1%; y decreases by 3%; z increases by 2%.

S: Note that $\triangle x = 0.01x$, $\triangle y = -0.03y$, $\triangle z = 0.02z$. $(*)$

Linear approximation:

$$\triangle w \approx \frac{\partial w}{\partial x}\triangle x + \frac{\partial w}{\partial y}\triangle y + \frac{\partial w}{\partial z}\triangle z = \frac{2\,xy^4}{z^3}\triangle x + \frac{4\,x^2y^3}{z^3}\triangle y - \frac{3\,x^2y^4}{z^4}\triangle z.$$

Use $\triangle x$, $\triangle y$, $\triangle z$ from $(*)$:

$$\triangle w \approx 2\underbrace{\frac{x^2y^4}{z^3}}_{=w}(0.01) + 4\underbrace{\frac{x^2y^4}{z^3}}_{=w}(-0.03) - 3\underbrace{\frac{x^2y^4}{z^3}}_{=w}(0.02)$$

$$= w\left[2\,(0.01) + 4\,(-0.03) - 3\,(0.02)\right],$$

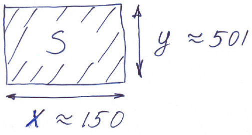
$$\Rightarrow \triangle w \approx w \cdot (-0.16) \quad \Rightarrow \frac{\triangle w}{w} \approx -0.16,$$

$$\Rightarrow \boxed{w \text{ decreases by about } 16\%}$$

§27.3 Applications of Linear Approximations to Error Analysis (2 Examples)

1. Two sides of a rectangular land are measured to be 150m and 501m. The length measurements were accurate to within 0.5m. What is the approximate maximum error if the area is calculated from these measurements?

<u>S:</u>



$x = 150 + \triangle x, \quad y = 501 + \triangle y$

with $|\triangle x| \leqslant 0.5$ and $|\triangle y| \leqslant 0.5$,

$S(x,y) = x \cdot y \quad \Rightarrow \frac{\partial S}{\partial x} = y, \quad \frac{\partial S}{\partial y} = x.$

Linear approximation near $(150, 501)$:

$$\underbrace{S(x,y) - S(150,501)}_{\triangle S = ??} \approx \frac{\partial S}{\partial x}\Big|_{(150,501)} \cdot \triangle x + \frac{\partial S}{\partial y}\Big|_{(150,501)} \cdot \triangle y$$

$$= 501 \cdot \triangle x + 150 \cdot \triangle y.$$

So for the error in the area we get: $\triangle S \approx 501 \cdot \triangle x + 150 \cdot \triangle y$,

$|\triangle S| \lesssim 501 \cdot |\triangle x| + 150 \cdot |\triangle y| \leqslant 501 \cdot 0.5 + 150 \cdot 0.5 = \boxed{325.5}$ —<u>Answer</u>.

NOTE: the calculated area $S(150, 501) = 150 \cdot 501 = 75151$,
so the error is small <u>relative</u> to the area...

2. The radius r and the height h of a cylinder are measured with a relative error of $\pm 1\%$. Find the relative error in its volume $V = \pi r^2 h$.

Remark: If a quantity Q is measured experimentally, then

Absolute Error $= Q_{\text{measured}} - Q_{\text{exact}}$	RelativeError $= \dfrac{\text{Abs.Error}}{Q_{\text{exact}}} \approx \dfrac{\text{Abs.Error}}{Q_{\text{measured}}}$

S:	Radius	Height	Volume
exact	r	h	$V(r,h) = \pi r^2 h$
measured	$r + \triangle r$	$h + \triangle h$	$V(r+\triangle r, h+\triangle h)$
absolute error	$\triangle r$	$\triangle h$	$\triangle V = V(r+\triangle r, h+\triangle h) - V(r,h)$
relative error	$\frac{\triangle r}{r}$	$\frac{\triangle h}{h}$	$\frac{\triangle V}{V}$

We know that $\left|\frac{\triangle r}{r}\right| \leqslant 0.01$ and $\left|\frac{\triangle h}{h}\right| \leqslant 0.01$, and need to estimate $\left|\frac{\triangle V}{V}\right|$.

Linear approximation of the function $V(r,h) = \pi r^2 h$:

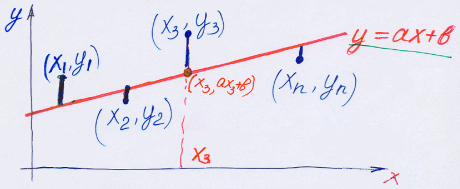
$\triangle V \approx \underbrace{2\pi r h}_{=\partial V/\partial r} \cdot \triangle r + \underbrace{\pi r^2}_{=\partial V/\partial h} \cdot \triangle h$ (exercise!) So $\frac{\triangle V}{V} = \frac{\triangle V}{\pi r^2 h} \approx 2\frac{\triangle r}{r} + \frac{\triangle h}{h}$

$$\Rightarrow \left|\frac{\triangle V}{V}\right| \lesssim 2\left|\frac{\triangle r}{r}\right| + \left|\frac{\triangle h}{h}\right| \leqslant 2 \cdot 0.01 + 0.01 = \boxed{0.03} \text{ or } 3\%.$$

NOTE: the volume V is twice as sensitive to error in r than to errors in h.

Lecture 28 The Method of Least Squares

An experiment to relate a quantity y to a quantity x, yields a set of data points $\boxed{(x_i, y_i) \text{ for } i = 1, 2, \cdots, n}$. Suppose, it is suspected that (if the measurements were perfect, but they never are) these points should lie on the same straight line $y = \mathbf{a}\,x + \mathbf{b}$:



Our Task: find the "best" line for our data set

(or, equivalently, find the "best" **a** and **b**).

For each x_i: the difference between the measured y-value y_i and the y-value on the line $a\,x_i + b$ is given by $\boxed{|a\,x_i + b - y_i|}$.

Method of Least Squares

Choose **a** and **b** to minimize the sum of the squares

$$S = \sum_{i=1}^{n} (a\,x_i + b - y_i)^2 \qquad (*)$$

To solve this problem: NOTE that here S is a function of two variables $S = S(a, b)$ (everything else is given data!),

i.e. $(*)$ is a minimization problem, only in 2 variables.

. . . .

Recall that $y = f(x)$ can have an extreme value at $x = x_0$

only if $\left.\frac{df}{dx}\right|_{x=x_0} = 0$.

Similarly, $z = f(x, y)$ can have an extreme value at (x_0, y_0)

only if $\boxed{\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = 0 \text{ and } \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = 0}$

. . . .

Apply this to our Minimization Problem $(*)$ by equating $\frac{\partial S}{\partial a}$ and $\frac{\partial S}{\partial b}$ to 0:

$$\frac{\partial S}{\partial a} = \sum_{i=1}^{n} 2\,(a\,x_i + b - y_i)\cdot x_i = 2\left(a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i \cdot y_i\right) = 0,$$

$$\frac{\partial S}{\partial b} = \sum_{i=1}^{n} 2\,(a\,x_i + b - y_i) = 2\left(a\sum_{i=1}^{n} x_i + b\,n - \sum_{i=1}^{n} y_i\right) = 0,$$

—i.e. we got 2 equations for a and b, and it remains to solve them.

—i.e. we got 2 equations for a and b, and it remains to solve them:

$$a\sum x_i^2 + b\sum x_i - \sum x_i \cdot y_i = 0 \quad (1)$$
$$a\sum x_i + b\,n - \sum y_i = 0 \quad (2)$$

To solve this system:

multiply (1) by n, then substitute bn obtained from (2):

$$a\,n\sum x_i^2 + \underbrace{b\,n}_{\text{from }(2)} \sum x_i - n\sum x_i \cdot y_i = 0,$$

$$a\,n\sum x_i^2 + \left[\sum y_i - a\sum x_i\right]\left(\sum x_i\right) - n\sum x_i \cdot y_i = 0.$$

Finally, for the Least Squares Method we get:

$$\boxed{\begin{aligned} a &= \frac{n\sum_{i=1}^n x_i \cdot y_i - \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right)}{n\sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2} \\ b &= \frac{1}{n}\left[\sum_{i=1}^n y_i - a\sum_{i=1}^n x_i\right] \end{aligned}} \quad (**)$$

NOTE: this method is also called Linear Regression,
while the line $y = ax + b$ is called the Regression Line.

Examples:

1. Find the least squares line for the data set:

$(0,2), (1,6), (2,4), (3,8), (4,10)$.

Solution: $n = 5$,

i	x_i	y_i	x_i^2	$x_i y_i$
1	0	2	0	0
2	1	6	1	6
3	2	4	4	8
4	3	8	9	24
5	4	10	16	40
$\sum$	10	30	30	78

$$a = \frac{n\sum x_i \cdot y_i - \left(\sum x_i\right)\left(\sum y_i\right)}{n\sum x_i^2 - \left(\sum x_i\right)^2} = \frac{(5)(78) - (10)(30)}{(5)(30) - (10)^2} = \frac{90}{50} = 1.8,$$

$$b = \frac{1}{n}\left[\sum y_i - a\sum x_i\right] = \frac{1}{5}\left[30 - 1.8 \cdot 10\right] = \frac{12}{5} = 2.4.$$

Answer: $\boxed{y = 1.8x + 2.4}$.

2. The least squares line for the data set: $(0,2)$, $(1,3)$, $(2,\bar{a})$, $(3,\bar{b})$, $(4,7)$ is $y = 2 + \frac{3}{2}x$. Find $\bar{a}$ and $\bar{b}$.

Solution: $n = 5$,

i	x_i	y_i	x_i^2	$x_i y_i$
1	0	2	0	0
2	1	3	1	3
3	2	$\bar{a}$	4	$2\bar{a}$
4	3	$\bar{b}$	9	$3\bar{b}$
5	4	7	16	28
$\sum$	10	$12+\bar{a}+\bar{b}$	30	$31+2\bar{a}+3\bar{b}$

$$\underbrace{\frac{3}{2}}_{=a} = \frac{n\sum x_i \cdot y_i - \left(\sum x_i\right)\left(\sum y_i\right)}{n\sum x_i^2 - \left(\sum x_i\right)^2} = \frac{(5)\,(31+2\bar{a}+3\bar{b})-(10)\,(12+\bar{a}+\bar{b})}{(5)\,(30)-(10)^2}$$

$$= \frac{35+5\bar{b}}{50} = \frac{7+\bar{b}}{10} \quad \Rightarrow \boxed{\bar{b} = 8}$$

$$\underbrace{2}_{=b} = \frac{1}{n}\left[\sum y_i - a\sum x_i\right] = \frac{1}{5}\left[(12+\bar{a}+\underbrace{\bar{b}}_{=8}) - \frac{3}{2}\cdot 10\right]$$

$$\Rightarrow\ 10 = (12+\bar{a}+8) - 15 = 5+\bar{a} \quad \Rightarrow \boxed{\bar{a} = 5}$$

Lecture 29 Introduction to Matrices (revision)

Definition

An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

The generic entry is a_{ij} (row i and column j).

E.g.: $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, where $4 = a_{12}$ and $3 = a_{31}$.

- If $m = n$, then A is a square matrix.
- An $1 \times n$ matrix is a row vector.

  E.g., $b = \begin{bmatrix} 1 & 4 & 8 \end{bmatrix}$ is a 1×3 row vector.
- An $m \times 1$ matrix is a column vector.

  E.g., $x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is a 2×1 column vector.

- Two matrices are called equal if they have the same size and equal corresponding entries.

$$\text{E.g.,}\quad \begin{bmatrix} 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \neq \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}.$$

- The $m \times n$ zero matrix is

$$A = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{m \times n}.$$

Notation: $A = 0$.

Matrix Addition

Matrices of the same size may be added: $\boxed{(A+B)_{ij} = a_{ij} + b_{ij}}$.

E.g.: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 6 & 5 \end{bmatrix}$.

Scalar Multiplication

Any $n \times m$ matrix may be multiplied by any real k: $\boxed{(k\,A)_{ij} = k\,a_{ij}}$.

E.g.: $-3 \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 6 & 0 \end{bmatrix}$;

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + 2 \begin{bmatrix} \frac{1}{2} & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 5 \\ 4 & 3 & 6 \end{bmatrix}.$$

Matrix Multiplication

One can multiply an $m \times n$ matrix A by an $n \times p$ matrix B:

then AB is an $m \times p$ matrix with $\boxed{(AB)_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}}$.

NOTE: one may expect $(AB)_{ij} = a_{ij}\, b_{ij}$ —WRONG!

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

$$\boxed{\underbrace{A}_{m \times n}\underbrace{B}_{n \times p} = \underbrace{C}_{m \times p}}$$

where

$$\boxed{c_{ij} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \cdots + a_{in}\, b_{nj}}$$

$=$ the dot-product of row i in A by column j in B.

Examples:

1. $$\underbrace{\begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -1 & 1 \\ 1 & 3 \end{bmatrix}}_{4\times 2} \cdot \underbrace{\begin{bmatrix} 0 & 2 & 4 \\ 1 & 3 & -2 \end{bmatrix}}_{2\times 3} = \underbrace{C}_{4\times 3} = ??$$

Answer: $C = \begin{bmatrix} 1 & 9 & 10 \\ 2 & 6 & -4 \\ 1 & 1 & -6 \\ 3 & 11 & -2 \end{bmatrix}$

$$c_{11} = R_1 C_1 = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \cdot 0 + 1 \cdot 1 = 1,$$

$$c_{12} = R_1 C_2 = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 3 \cdot 2 + 1 \cdot 3 = 9,$$

$$c_{13} = R_1 C_3 = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 3 \cdot 4 + 1 \cdot (-2) = 10,$$

$$c_{21} = R_2 C_1 = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot 0 + 2 \cdot 1 = 2...$$

2. $$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

NOTE: this example shows that $AB = 0$ does NOT imply that either $A = 0$ or $B = 0$.

NOTE also:

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \text{ gives } A^2 = A \cdot A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ 4 & 8 \end{bmatrix}.$$

Some Rules of Matrix Arithmetic

1. $(k\,A)\,B = A\,(k\,B) = k\,(A\,B)$
2. $\boxed{(A\,B)\,C = A\,(B\,C)}, \quad \neq (B\,C)\,A$
3. $(A + B)\,C = A\,C + B\,C, \quad \neq C\,A + C\,B$
4. $A\,(B + C) = A\,B + A\,C$
5. $A + 0 = A, \qquad A + (-A) = 0, \qquad A \cdot 0 = 0$

Remark: in general, $\boxed{A\,B \neq B\,A}$

Matrix Transpose

The transpose of an $n \times m$ matrix A is the $m \times n$ matrix A^T such that

$$\boxed{\left(A^T\right)_{ij} = a_{ji}}.$$

—i.e. the rows of A become columns of A^T

(or, equivalently, the columns of A become rows of A^T).

E.g.:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix};$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$$

Properties: $\boxed{\left(A^T\right)^T = A}$ $\boxed{(A\,B)^T = B^T A^T}$ $\boxed{(k\,A)^T = k\,A^T}$

Identity Matrix I_n

is the $n \times n$ matrix with 1s on the diagonal and 0s elsewhere:

$$I_1 = [1];\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};\ I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ldots$$

NOTE: If A is $m \times n$, then $\boxed{I_m A = \underbrace{A}_{m\times n} = A\, I_n}$.

E.g.: $\begin{bmatrix} 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 2 \end{bmatrix}}_{1\times 2} = \begin{bmatrix} 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \end{bmatrix}$.

Matrix Inverse

Def: An $n \times n$ matrix B is an inverse for a square $n \times n$ matrix A if $\boxed{AB = BA = I_n}$.

Examples: **(1)** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ is an inverse of A, since $AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(2) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has NO inverse!!

Proof: Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an inverse of A.

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = AB = \begin{bmatrix} a+2c & b+2d \\ 2a+4c & 2b+4d \end{bmatrix}$$

$\Rightarrow a + 2c = 1$ and $2a + 4c = 0$ —impossible $\Rightarrow$ NO inverse! □

Theorem

If A has an inverse, it is unique. (without proof)

Notation: A^{-1} denotes the unique inverse of A; then A is called invertible.

Properties: $\boxed{(A^{-1})^{-1} = A}$ $\boxed{(AB)^{-1} = B^{-1}A^{-1}}$ $\boxed{(kA)^{-1} = \frac{1}{k}A^{-1} \; (k \neq 0)}$

Matrix Representation

A system of m linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

—This system may be rewritten as a single matrix equation

$$\underbrace{A}_{m\times n}\underbrace{x}_{n\times 1} = \underbrace{b}_{m\times 1}$$

as follows:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{A\,:\,m\times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x\,:\,n\times 1} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b\,:\,m\times 1}$$

Examples:

1. $$\begin{aligned} 4x_1 + 2x_2 &= 3 \\ x_2 &= 1 \end{aligned} \Rightarrow \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Remark:

It is convenient to skip x (as it contains no info) and instead use the augmented matrix $\left[\, A \mid b \,\right]$ —Here A is augmented by b.
Indeed the augmented matrix contains all the given data of the system!

So, for our example, we equivalently have $\left[\begin{array}{cc|c} 4 & 2 & 3 \\ 0 & 1 & 1 \end{array}\right]$

2. Trivial system:

$$\begin{aligned} x_1 \quad\quad &= 4 \\ x_2 \quad &= 3 \\ x_3 &= 2 \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=I_3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

3. $$\begin{aligned} x_1 + x_2 + \quad + x_4 &= 1 \\ x_2 + x_3 + x_4 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

or, equivalently, $\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]$

§30.2 How Many Solutions?

Consider **2 unknowns**: $x_1 = x$ and $x_2 = y$.

Each equation in the system has the form $\boxed{a\,x + b\,y = c}$

—this is an equation of a **line** on the plane!

Suppose we have **2 equations** in **2 unknowns**:

$\boxed{a_{11}\,x + a_{12}\,y = b_1}$ —this is an equation of a line L_1 on the plane;

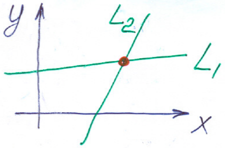
$\boxed{a_{21}\,x + a_{22}\,y = b_2}$ —this is an equation of a line L_2 on the plane.

So, any solution is a point that belongs to both lines;

i.e. any solution is an intersection of the 2 lines!

Hence, we have 3 possibilities:

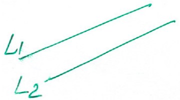
(a) The lines L_1 and L_2 are not parallel, then they intersect at one point:



$\Rightarrow$ $\boxed{\text{one solution}}$ (consistent system)

E.g.: $\begin{array}{l} x + y = 1 \\ x - y = 1 \end{array}$ has the unique solution $\begin{array}{l} x = 1, \\ y = 0. \end{array}$

(b) The lines L_1 and L_2 are parallel, then they do NOT intersect:



$\Rightarrow$ NO solutions (inconsistent system)

E.g.: the system $\begin{array}{l} x + y = 1 \\ x + y = 2 \end{array}$ has no solutions.

(c) The lines L_1 and L_2 coincide, the intersection is the entire line:

$\Rightarrow$ ∞ solutions (consistent system),
all the points on the line are solutions.

E.g.: $\begin{array}{r} x - y = 1 \\ 2x - 2y = 2 \end{array}$ has ∞ solutions $\begin{bmatrix} x \\ x - 1 \end{bmatrix}$.

Consider **3 unknowns**: $x_1 = x$, $x_2 = y$ and $x_3 = z$.
Each equation in the system has the form $ax + by + cz = d$

—this is an equation of a **plane** in the space!

Any solution for **3 equations** in **3 unknowns** is an intersection of the 3 planes: this may be (i) 1 point (so 1 solution); (ii) a line (∞ solutions); (iii) a plane (∞ solutions); (iv) NO intersections (NO solutions).

Consider **more unknowns** (higher dimensions):

one has to resort to algebraic analysis...

§30.3 Gauss-Jordan Elimination

This method solves any linear system, i.e. **(1)** detects whether the system is consistent; **(2)** if it is, then the method yields ALL solutions.

Elementary Row Operations

(i) Interchange any 2 equations (rows): $\boxed{R_i \leftrightarrow R_j}$.

(ii) Multiply any equation (row) by a nonzero constant: $\boxed{k\,R_i \;\; (k \neq 0)}$.

(iii) Add a multiple of one equation (row) by any constant to another: $\boxed{R_i + k\,R_j}$.

NOTE: here rows refer to rows of the augmented matrix.

If any elementary row operation is applied to the augmented matrix $\left[\,A \,\middle|\, b\,\right]$, the resulting matrix has the same set of solutions.

$\Rightarrow$ OUR PLAN: apply elementary row operations to reduce the augmented matrix to an equivalent simple form!

Gauss-Jordan Elimination: apply elementary row operations to reduce the augmented matrix to its $\boxed{\text{RREF}}$ — Reduced Row Echelon Form.

RREF — Reduced Row Echelon Form

The first nonzero element in each nonzero row is the only nonzero entry in its column.

E.g.: $\begin{bmatrix} \boxed{1} & 0 & 0 & 1 & 3 \\ 0 & 0 & \boxed{2} & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (YES); $\begin{bmatrix} \boxed{1} & 0 & 1 & 1 & 3 \\ 0 & 0 & \boxed{2} & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (NO).

NOTE: the first nonzero element in a row is called the pivot for that row.

. . . .

Examples: Find the RREF for each system and hence solve it.

1. $$\begin{aligned} x_1 + 2x_2 + x_3 &= 6 \\ 2x_1 \qquad - x_3 &= 1 \\ x_2 + 2x_3 &= 4 \end{aligned}$$ $\Rightarrow$ Augmented matrix: $\left[\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 6 \\ \underline{2} & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right]$

—here $\boxed{1}$ is the pivot for row R_1.

—Use elementary row operations to transform the other entries in column 1 to zeros (i.e. to transform $\underline{2}$ to 0): $\boxed{R_2 - 2\,R_1}$ yields $\Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -4 & -3 & -11 \\ 0 & \boxed{1} & 2 & 4 \end{array}\right]$$

$\Rightarrow$To simplify calculations: $\boxed{-R_2}$, and also $\boxed{R_2 \leftrightarrow R_3}$ (to use $\boxed{1}$ instead of -4 as a pivot):

$$\left[\begin{array}{ccc|c} 1 & \underline{2} & 1 & 6 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & \underline{4} & 3 & 11 \end{array}\right]$$

$\Rightarrow$To transform $\underline{2}$ and $\underline{4}$ to zeros, apply $\boxed{R_1 - 2\,R_2}$ and $\boxed{R_3 - 4\,R_2}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & \underline{-3} & -2 \\ 0 & 1 & \underline{2} & 4 \\ 0 & 0 & \boxed{-5} & -5 \end{array}\right]$$

$\Rightarrow$To simplify calculations: $\boxed{-\frac{1}{5}\,R_3}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & \underline{-3} & -2 \\ 0 & 1 & \underline{2} & 4 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

$\Rightarrow$To transform $\underline{-3}$ and $\underline{2}$ to zeros, apply $\boxed{R_1 + 3\,R_3}$ and $\boxed{R_2 - 2\,R_3}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

—this is RREF; it yields a trivial system: $\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 1 \end{cases}$

Answer: $(x_1,\, x_2,\, x_3) = (1,\, 2,\, 1)$.

2 $\left[\begin{array}{cccc|c} 0 & 0 & \boxed{1} & 1 & 2 \\ 1 & 0 & 1 & -1 & 0 \\ 1 & 2 & -2 & 0 & 2 \\ 1 & 1 & 0 & 0 & 2 \end{array}\right].$

S: Pivot on each row using elementary row operations.

First, $\boxed{R_2 - R_1}$ and $\boxed{R_3 + 2R_1}$ yield: $\left[\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 2 \\ \boxed{1} & 0 & 0 & -2 & -2 \\ 1 & 2 & 0 & 2 & 6 \\ 1 & 1 & 0 & 0 & 2 \end{array}\right]$ —done with Row 1.

Next, $\boxed{R_3 - R_2}$ and $\boxed{R_4 - R_2}$ yield: $\left[\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & -2 & -2 \\ 0 & \boxed{2} & 0 & 4 & 8 \\ 0 & 1 & 0 & 2 & 4 \end{array}\right]$ —done with Row 2.

To simplify calculations $\boxed{\frac{1}{2} R_3}$ $\Rightarrow$ $\left[\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & -2 & -2 \\ 0 & \boxed{1} & 0 & 2 & 4 \\ 0 & 1 & 0 & 2 & 4 \end{array}\right].$

Now, $\boxed{R_4 - R_3}$ yields: $\left[\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

—Here the final row is equivalent to $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 0$, i.e. says NOTHING! $\Rightarrow$ delete it!

Finally, reorder the rows: $\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & -2 & -2 \\ 0 & \boxed{1} & 0 & 2 & 4 \\ 0 & 0 & \boxed{1} & 1 & 2 \end{array}\right]$ —this is RREF!

To complete the solution:

Note that the remaining 3 rows involve the 3 pivots $\boxed{1}$ that correspond to the 3 unknowns x_1, x_2 and x_3.

The remaining unknown x_4 is called a free variable.

Rewrite the RREF as a system:

$$\begin{array}{rcl} x_1 \qquad\qquad - 2\,x_4 & = & -2 \\ x_2 \qquad + 2\,x_4 & = & 4 \\ x_3 + x_4 & = & 2 \end{array}$$

Set $\boxed{x_4 = t}$ —any real number.

$\Rightarrow$ Answer: $\boxed{(x_1,\, x_2,\, x_3,\, x_4) = (-2 + 2t,\, 4 - 2t,\, 2 - t,\, t), \quad t \in \mathbb{R}}$
(i.e. the system is consistent and we have infinitely many solutions).

NOTE: using the Answer, we can get particular solutions,
e.g., $t = 0 \Rightarrow (-2, 4, 2, 0)$ and $t = 1 \Rightarrow (0, 2, 1, 1)$.

Remark

A row $\left[\begin{array}{cccc|c} 0 & 0 & 0\cdots & 0 & a \end{array}\right]$ with $a \neq 0$, is equivalent to the equation

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \cdots + 0 \cdot x_n = a \neq 0.$$

So the system is inconsistent (i.e. has NO solutions).

3. $$\begin{aligned} 2x_1 + x_2 &= 1 \\ x_1 + 3x_2 &= -7 \\ x_1 + 2x_2 &= -3 \end{aligned} \Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 3 & -7 \\ 1 & 2 & -3 \end{array}\right]$$ First, $\boxed{R_1 \leftrightarrow R_2}$ (to avoid fractions)

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 3 & -7 \\ 2 & 1 & 1 \\ 1 & 2 & -3 \end{array}\right].$$ So use $\boxed{R_2 - 2R_1}$, $\boxed{R_3 - R_1}$ $$\Rightarrow \left[\begin{array}{cc|c} 1 & 3 & -7 \\ 0 & -5 & 15 \\ 0 & -1 & 4 \end{array}\right]$$

$$\boxed{-\tfrac{1}{5} R_2} \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & -7 \\ 0 & 1 & -3 \\ 0 & -1 & 4 \end{array}\right].$$ Next, $\boxed{R_1 - 3R_2}$, $\boxed{R_3 + R_2}$ $$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array}\right]$$

—this is the RREF, in which Row 3 gives: $0 \cdot x_1 + 0 \cdot x_2 = 1$.

$\Rightarrow$ Answer: the system is inconsistent (i.e. NO solutions).

Remark

For any $m \times n$ system, there are 3 possibilities:
(i) a unique solution; **(ii)** NO solutions; **(iii)** ∞ solutions.

E.g., it's impossible for a system to have just 2 solutions...

Further Examples:

1. $x_1 + 2x_2 - 3x_3 = 5 \quad \Rightarrow \left[\begin{array}{ccc|c} \boxed{1} & 2 & -3 & 5 \end{array}\right].$

The pivot $\boxed{1}$ is for x_1. The remaining x_2 and x_3 will be free variables.

Set $\boxed{x_2 = s, \; x_3 = t}$ $\Rightarrow$ Answer: $\boxed{(x_1, x_2, x_3) = (5 - 2s + 3t, \; s, \; t), \; s, t \in \mathbb{R}}$

2. $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 1 \\ -1 & -1 & 1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 1 \end{array}\right] \Rightarrow \underbrace{\cdots}_{\text{Exercise}} \Rightarrow \left[\begin{array}{cccc|c} \boxed{1} & 0 & -5 & -5 & -1 \\ 0 & \boxed{1} & 4 & 3 & 1 \end{array}\right]$

The 2 pivots $\boxed{1}$ are for x_1, x_2. The remaining x_3, x_4 are free variables.

Set $\boxed{x_3 = t, \;\; x_4 = s}$ $\Rightarrow$

Answer: $\boxed{(x_1, x_2, x_3, x_4) = (-1 + 5t + 5s, \; 1 - 4t - 3s, \; t, \; s), \; t, s \in \mathbb{R}}$

Lecture 31 Inverse of a Square Matrix by the Gauss-Jordan Elimination

IDEA of the method: to find A^{-1} we need to solve the matrix equation $AX = \underbrace{I}_{\text{identity matrix}}$. Then the solution $X = A^{-1}$.

E.g.: for a 3×3 matrix A: $A \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

In fact, this matrix equation is equivalent to 3 systems:

$$\textbf{(i)}\ A \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};\ \textbf{(ii)}\ A \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};\ \textbf{(iii)}\ A \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

For those 3 systems, the augmented matrices are:

$$\textbf{(i)} \left[\begin{array}{c|c} & 1 \\ A & 0 \\ & 0 \end{array}\right];\quad \textbf{(ii)} \left[\begin{array}{c|c} & 0 \\ A & 1 \\ & 0 \end{array}\right];\quad \textbf{(iii)} \left[\begin{array}{c|c} & 0 \\ A & 0 \\ & 1 \end{array}\right].$$

NOTE: If A is invertible, then (without proof, see further examples) the RREF for the system $\left[\, A \mid b \,\right]$ (with any right-hand side vector b) has the form $\left[\, I \mid c \,\right]$ for some vector c. Consequently, the unique solution of this system is $x_1 = c_1,\ x_2 = c_2,\ x_3 = c_3$.

Apply this observation to our 3 systems: their RREF will be

$$\textbf{(i)} \left[\begin{array}{c|c} & x_{11} \\ I & x_{12} \\ & x_{13} \end{array}\right]; \quad \textbf{(ii)} \left[\begin{array}{c|c} & x_{21} \\ I & x_{22} \\ & x_{23} \end{array}\right]; \quad \textbf{(iii)} \left[\begin{array}{c|c} & x_{31} \\ I & x_{32} \\ & x_{33} \end{array}\right].$$

It is convenient to combine the 3 systems and transform them to the RREF together as follows:

Computation of A^{-1} (Description of the Method)

- Form the augmented matrix $\boxed{\left[\, A \mid I \,\right]}$.
- Use the Gauss-Jordan elimination to transform it to the RREF $\boxed{\left[\, I \mid X \,\right]}$.
- If the reduction can be carried out, then $\boxed{A^{-1} = X}$. Otherwise, A^{-1} does NOT exist.

Examples:

1. $A = \begin{bmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 1 & 5 & -2 \end{bmatrix}$ S: $[\,A \mid I\,] = \left[\begin{array}{ccc|ccc} \boxed{1} & 3 & -1 & 1 & 0 & 0 \\ \underline{-2} & -5 & 1 & 0 & 1 & 0 \\ \underline{1} & 5 & -2 & 0 & 0 & 1 \end{array}\right]$

Apply $\boxed{R_2 + 2\,R_1}$ and $\boxed{R_3 - R_1}$ $\Rightarrow \left[\begin{array}{ccc|ccc} 1 & \underline{3} & -1 & 1 & 0 & 0 \\ 0 & \boxed{1} & -1 & 2 & 1 & 0 \\ 0 & \underline{2} & -1 & -1 & 0 & 1 \end{array}\right]$

Apply $\boxed{R_1 - 3\,R_2}$ and $\boxed{R_3 - 2\,R_2}$ $\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & \underline{2} & -5 & -3 & 0 \\ 0 & 1 & \underline{-1} & 2 & 1 & 0 \\ 0 & 0 & \boxed{1} & -5 & -2 & 1 \end{array}\right]$

Apply $\boxed{R_1 - 2\,R_3}$ and $\boxed{R_2 + R_3}$ $\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 1 & -2 \\ 0 & 1 & 0 & -3 & -1 & 1 \\ 0 & 0 & 1 & -5 & -2 & 1 \end{array}\right]$

$\Rightarrow$ Answer: $\begin{bmatrix} 5 & 1 & -2 \\ -3 & -1 & 1 \\ -5 & -2 & 1 \end{bmatrix}$.

To check the Answer: check that $A\,A^{-1} = I$!

2. $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & 2 \\ 2 & -5 & -1 \end{bmatrix}$

S: $\left[\, A \mid I \,\right] = \left[\begin{array}{ccc|ccc} \boxed{1} & -2 & 1 & 1 & 0 & 0 \\ -1 & 3 & 2 & 0 & 1 & 0 \\ 2 & -5 & -1 & 0 & 0 & 1 \end{array}\right]$

Apply $\boxed{R_2 + R_1}$ and $\boxed{R_3 - 2\,R_1}$ $\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 3 & 1 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right]$

Apply $\boxed{R_1 + 2\,R_2}$ and $\boxed{R_3 + R_2}$ $\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 7 & 3 & 2 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$

$\Rightarrow$ Answer: A is NOT invertible (i.e. A^{-1} does NOT exist).

3. $A = \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix}$ S: $[A \mid I] = \left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 2 & \boxed{1} & 0 & 1 \end{array}\right]$

Apply $\boxed{R_1 - 5R_2}$ $\Rightarrow \left[\begin{array}{cc|cc} \boxed{-7} & 0 & 1 & -5 \\ 2 & 1 & 0 & 1 \end{array}\right]$

Apply $\boxed{2R_1}$ and $\boxed{7R_2}$ (to avoid fractions) $\Rightarrow \left[\begin{array}{cc|cc} \boxed{-14} & 0 & 2 & -10 \\ 14 & 7 & 0 & 7 \end{array}\right]$

Apply $\boxed{R_2 + R_1}$ $\Rightarrow \left[\begin{array}{cc|cc} -14 & 0 & 2 & -10 \\ 0 & 7 & 2 & -3 \end{array}\right]$

Apply $\boxed{-\frac{1}{14}R_1}$ and $\boxed{\frac{1}{7}R_2}$ $\Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{7} & \frac{5}{7} \\ 0 & 1 & \frac{2}{7} & -\frac{3}{7} \end{array}\right]$

Answer: $A^{-1} = \begin{bmatrix} -\frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$.

Verify: $AA^{-1} = \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4.

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 3 & 1 & 4 \\ 2 & 7 & 6 & -1 \\ 1 & 2 & 2 & -1 \end{bmatrix}. \quad \underline{\text{S:}} \ [\, A \mid I \,] = \left[\begin{array}{cccc|cccc} 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 4 & 0 & 1 & 0 & 0 \\ 2 & 7 & 6 & -1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \underbrace{\cdots}_{\text{Handout}} \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{2} & -\frac{7}{6} & \frac{10}{3} \\ 0 & 1 & 0 & 0 & -\frac{7}{6} & -\frac{1}{2} & \frac{5}{6} & -\frac{5}{3} \\ 0 & 0 & 1 & 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right].$$

$$\Rightarrow \underline{\text{Answer:}} \ A^{-1} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{2} & -\frac{7}{6} & \frac{10}{3} \\ -\frac{7}{6} & -\frac{1}{2} & \frac{5}{6} & -\frac{5}{3} \\ \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}.$$

Definition by induction

Let A be an $n \times n$ matrix. The determinant of A,
denoted $\boxed{\det A}$ or $\boxed{|A|}$, is the number defined as follows.

n = 1: i.e. $A = [a_{11}]$. Then $\boxed{\det[a_{11}] = |a_{11}| = a_{11}}$.

E.g.: $\det[5] = |5| = 5$, $\det[-1] = |-1| = -1$

(do NOT confuse with the absolute value!)

$$\mathbf{n} = \mathbf{2}: \quad \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \boxed{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}}.$$

Remark: this definition may be interpreted as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot \det\left[a_{22}\right] - a_{12} \cdot \det\left[a_{21}\right]$$ —First-Row Expansion.

E.g.: $\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2 \cdot 3 - 4 \cdot 1 = 2.$

$$\mathbf{n = 3}: \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$= \boxed{a_{11} \cdot \underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{R_1,\, C_1 \text{ deleted in } A} - a_{12} \cdot \underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{R_1,\, C_2 \text{ deleted in } A} + a_{13} \cdot \underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{R_1,\, C_3 \text{ deleted in } A}}.$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

—First-Row Expansion!

E.g.: $A = \begin{bmatrix} 1 & 2 & -1 \\ 5 & 3 & 4 \\ -2 & 0 & 1 \end{bmatrix}$,

$$\det A = 1 \cdot \underbrace{\begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix}}_{=3\cdot 1-4\cdot 0} - 2 \cdot \underbrace{\begin{vmatrix} 5 & 4 \\ -2 & 1 \end{vmatrix}}_{=5\cdot 1-4\cdot(-2)} + (-1) \cdot \underbrace{\begin{vmatrix} 5 & 3 \\ -2 & 0 \end{vmatrix}}_{=5\cdot 0-3\cdot(-2)}$$

$$= 1 \cdot 3 - 2 \cdot 13 + (-1) \cdot 6 = -29. \qquad \square$$

n = 4 :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \cdot \underbrace{\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}}_{R_1,\ C_1 \text{ deleted}} - a_{12} \cdot \underbrace{\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}}_{R_1,\ C_2 \text{ deleted}}$$

$$+ a_{13} \cdot \underbrace{\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}}_{R_1,\ C_3 \text{ deleted}} - a_{14} \cdot \underbrace{\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}}_{R_1,\ C_4 \text{ deleted}}$$

—again the First-Row Expansion!

Here we used:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Example:

$$\begin{vmatrix} 1 & 2 & 0 & 2 \\ -1 & 2 & 3 & 1 \\ -3 & 2 & -1 & 0 \\ 2 & -3 & -2 & 1 \end{vmatrix} = 1 \cdot \underbrace{\begin{vmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & -2 & 1 \end{vmatrix}}_{=-15} - 2 \cdot \underbrace{\begin{vmatrix} -1 & 3 & 1 \\ -3 & -1 & 0 \\ 2 & -2 & 1 \end{vmatrix}}_{=18}$$

$$+ 0 \cdot \begin{vmatrix} -1 & 2 & 1 \\ -3 & 2 & 0 \\ 2 & -3 & 1 \end{vmatrix} - 2 \cdot \underbrace{\begin{vmatrix} -1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -3 & -2 \end{vmatrix}}_{=6}$$

Ex.: Check the 3×3 determinants!

$$= 1 \cdot (-15) - 2 \cdot (18) + 0 - 2 \cdot (6) = -63. \qquad \square$$

Similarly, any **n** $\times$ **n** determinant

is defined in terms of n determinants $(n-1) \times (n-1)$

via the First-Row Expansion...

(Another) EXAMPLE (∗):

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{bmatrix}$$

$$\det A = 1 \cdot \begin{vmatrix} \boxed{1} & 2 & -1 & 1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & \boxed{2} & -1 & 1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{vmatrix}$$

$$+ (-1) \cdot \begin{vmatrix} 1 & 2 & \boxed{-1} & 1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 2 & -1 & \boxed{1} \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{vmatrix},$$

$$\begin{vmatrix} 0 & 2 & -2 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 2; \begin{vmatrix} -1 & 2 & -2 \\ 3 & 1 & 1 \\ 2 & -1 & 2 \end{vmatrix} = -1; \begin{vmatrix} -1 & 0 & -2 \\ 3 & -1 & 1 \\ 2 & 0 & 2 \end{vmatrix} = -2; \begin{vmatrix} -1 & 0 & 2 \\ 3 & -1 & 1 \\ 2 & 0 & -1 \end{vmatrix} = 3$$

(Ex.: check this)

$$\det A = 1 \cdot 2 - 2 \cdot (-1) + (-1) \cdot (-2) - 1 \cdot 3 = 3. \quad \square$$

§32.2 Alternative Evaluation

Theorem: $\det A =$ "cofactor" expansion along any ROW or COLUMN.
(without proof).

Row r Expansion

$$\boxed{\det A = +a_{r1} \cdot \left|\cdots\right| - a_{r2} \cdots \left|\cdots\right| + \cdots \pm a_{rn}\left|\cdots\right|}$$ if r is odd,

$$\boxed{\det A = -a_{r1} \cdot \left|\cdots\right| + a_{r2} \cdots \left|\cdots\right| - \cdots \pm a_{rn}\left|\cdots\right|}$$ if r is even.

where each a_{rj} is multiplied by the $(n-1) \times (n-1)$ determinant (denoted $\left|\cdots\right|$), obtained by deleting row r and column j in the original matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Column s Expansion

For an $n \times n$ matrix A: $\det A = a_{1s} \cdot A_{1s} + a_{2s} \cdot A_{2s} + \cdots + a_{ns} \cdot A_{ns}$

$$\det A = +a_{1s} \cdot \left|\cdots\right| - a_{2s} \cdots \left|\cdots\right| + \cdots \pm a_{ns}\left|\cdots\right| \quad \text{if } s \text{ is odd,}$$

$$\det A = -a_{1s} \cdot \left|\cdots\right| + a_{2s} \cdots \left|\cdots\right| - \cdots \pm a_{ns}\left|\cdots\right| \quad \text{if } s \text{ is even.}$$

where each a_{is} is multiplied by the $(n-1) \times (n-1)$ determinant (denoted $\left|\cdots\right|$), obtained by deleting row i and column s in the original matrix A:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1s} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2s} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{ns} & \cdots & a_{nn} \end{bmatrix}.$$

NOTE the sign pattern:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(Earlier) EXAMPLE (*):

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{bmatrix}$$

$$\det A = -2 \cdot \begin{vmatrix} -1 & 2 & -2 \\ 3 & 1 & 1 \\ 2 & -1 & 2 \end{vmatrix} + 0 \cdot \begin{vmatrix} \cdots \end{vmatrix} - (-1) \cdot \begin{vmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 2 & -1 & 2 \end{vmatrix} + 0 \cdot \begin{vmatrix} \cdots \end{vmatrix},$$

where $\begin{vmatrix} -1 & 2 & -2 \\ 3 & 1 & 1 \\ 2 & -1 & 2 \end{vmatrix} = -1$ (see earlier); $\begin{vmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 2 & -1 & 2 \end{vmatrix} = 1$ (Ex.!!!);

$$\det A = -2 \cdot (-1) + 0 - (-1) \cdot 1 + 0 = 3.$$ —same result!

Another Example:

$$A = \begin{bmatrix} 1 & 2 & 3 & 7 \\ 4 & 0 & 8 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\det A = -2 \cdot \begin{vmatrix} 4 & 8 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{vmatrix} + 0 \cdot \begin{vmatrix} \cdots \end{vmatrix} - 0 \cdot \begin{vmatrix} \cdots \end{vmatrix} + 0 \cdot \begin{vmatrix} \cdots \end{vmatrix},$$

$$= -2 \cdot \begin{vmatrix} 4 & 8 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{vmatrix} = -2 \cdot \left(4 \cdot \begin{vmatrix} 1 & 5 \\ 0 & 3 \end{vmatrix} - 0 + 0 \right) = -24.$$

Warning: Watch that each term has the right sign $\pm$!!!

§32.3 Easy Determinants

1. If A has a row or column of zero entries, then $\boxed{\det A = 0}$.

E.g.: $\begin{vmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ -1 & 1 & 2 \end{vmatrix} = 0$, $\quad \begin{vmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 0 \end{vmatrix} = 0$.

2. If 2 rows or 2 columns are equal or proportional, then $\boxed{\det A = 0}$.

E.g.: $\underbrace{\begin{vmatrix} 1 & -1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 3 \end{vmatrix}}_{C_1 = C_3} = 0$, $\quad \underbrace{\begin{vmatrix} 2 & 4 & 6 \\ -1 & -2 & -3 \\ 1 & -1 & 2 \end{vmatrix}}_{R_1 = -2R_2} = 0$.

3. If A is:

either lower triangular, i.e.

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

—all entries above the diagonal $= 0$

or upper triangular, i.e.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

—all entries below the diagonal $= 0$

$\Rightarrow$ $\det A = a_{11} \cdot a_{22} \cdot a_{33} \cdots a_{nn} =$ product of the diagonal entries

E.g.: $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 18 & 1 & 0 \\ 11 & 101 & 8 & 5 \end{vmatrix} = (1) \cdot (-1) \cdot (1) \cdot (5) = -5.$

Proof: Use the first row expansion:

$$\Rightarrow \det A = (1) \cdot \underbrace{\begin{vmatrix} -1 & 0 & 0 \\ 18 & 1 & 0 \\ 101 & 8 & 5 \end{vmatrix}}_{\text{First Row Expansion}} = (1) \cdot (-1) \cdot \underbrace{\begin{vmatrix} 1 & 0 \\ 8 & 5 \end{vmatrix}}_{=(1)\cdot(5)}. \quad \square$$

Final Exam

- Check the MA4002 website at
  `http://www.staff.ul.ie/natalia/MA4002.html`

  —NOTE the Important Info file there.

- General Advice:
  2 examples may seem similar (replace 1 by 3...), but solutions may be quite different. If you target a particular question,

  be prepared to solve the entire CLASS of problems!

  (not just an example from the last year paper): i.e., be prepared to different scenarios in the solution process... To prepare for this, carefully check ALL examples and notes in the relevant lecture...

Thanks for Your Attention & Best of Luck!