MA4002 Engineering Mathematics 2

https://staff.ul.ie/natalia/node/1200

Prof. Natalia Kopteva

Spring 2023

- WELCOME to
  MA4002 Engineering Mathematics 2

- Your lecturer: Prof. Natalia Kopteva
  https://staff.ul.ie/natalia/

  – Have been full-time lecturing in Ireland + UK since 2000.

  – If you want to chat with me, I'll have office hours every week in my office B2032, the time slots t.b. confirmed via SULIS

  – or you may contact me via email at **natalia.kopteva@ul.ie** to ask a question.

- 3 live lectures each week (also prerecorded via PANOPTO in 2021)

- Tutorials: **one 2-hour slot per week starting from Week 2**

- Tutorials Sheets: are available via **SULIS only**.

- In addition to MA4002 SULIS website,
there is an open-access MA4002 website:
https://staff.ul.ie/natalia/node/1200
(this link is also given at the SULIS site, or you may Google it;
different from earlier years!!!)

+ Prime Text + past Midterm papers + Final exam papers with solutions

+ **Lecture Notes** will also be given at the above website, as well as will be available via SULIS.

+ a lot more...

- COURSE OUTLINE

  - See the first SULIS Handout for syllabus details

  - One central topic will be

    Integration + its Applications

- Philosophy:

  Education is what remains after one has forgotten what one has learned in school. Albert Einstein

  You'll forget the formulas and recipes. What is worth learning is HOW to APPROACH a problem. Only then, as an engineer, you'll be able to solve non-standard and non-trivial problems...

  – NOT (not only) memorize a number of formulas,
  but UNDERSTAND where these formulas come from...

  (Our objectives may be quite different from school maths

  – The world is changing very rapidly. In future, you may have to move to entirely different professions. To be able to compete, one needs to be able to solve non-standard problems, think outside the box. This comes from a true understanding of things, as recipes can be implemented by the machines much more efficiently...

- Example: We'll study Antiderivatives and Integrals "from scratch" (as if you didn't do them at school).

  Why? – At school, you studied a bunch of recipes;
  we try to understand where they come from...

- A bit of ADVICE (more to follow):

  – TAKE some NOTES by hand
  (during lectures / listening to recordings + preparing for tutorials/exams)
  https://www.bbc.com/future/article/20191122-when-the-best-way-to-take-notes-is-by-hand

  – do ATTEMPT to solve tutorial problems BEFORE reading provided TUTORIAL solutions (not going to be easy, but VERY beneficial...)

  – ASK QUESTIONS: during l tutorials / office hours, afterwards...

  – look for HELP if you find the module difficult (well before the exams)

  – STUDY GROUPS: for many students, it's beneficial to study maths in groups...

  – target to work **8-10 HOURS per week** (unless you find the module easy)

NEXT:

"Forget"

"FORGET" all you know from school about
Antiderivatives + INTEGRATION

Now

START from SCRATCH

(Why? – At school, you studied a bunch of recipes;
we try to understand where they come from...)

Lecture 1: §1.1 Antiderivatives

Definition

An antiderivative of a function f on an interval I is a function F that satisfies $\boxed{F'(x) = f(x) \text{ for all } x \in I}$

Examples

1. $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$ since $\frac{d}{dx}\left(\frac{1}{2}x^2\right) = x$.

2. $F(x) = -\frac{1}{4}\cos(4x)$ is an antiderivative of $f(x) = \sin(4x)$ since $\frac{d}{dx}\left(-\frac{1}{4}\cos(4x)\right) = \sin(4x)$.

   So is $-\frac{1}{4}\cos(4x) + 5$ since $\frac{d}{dx}\left(-\frac{1}{4}\cos(4x) + 5\right) = \sin(4x)$.

   So is $-\frac{1}{4}\cos(4x) + C$ for any real constant C, since $\frac{d}{dx}\left(-\frac{1}{4}\cos(4x) + C\right) = \sin(4x)$.

   (NOTE: this shows that antiderivatives are not unique.)

§1.2 The Indefinite Integral

Let $F(x)$ be any particular antiderivative of $f(x)$.

Then the general antiderivative of $f(x)$ is $\boxed{F(x) + C}$ (also called the indefinite integral):

Definition

The indefinite integral of a function f on an interval I is

$\boxed{\int f(x)\,dx = F(x) + C \text{ on } I}$ where $F'(x) = f(x)$ for all $x \in I$.

Here:

$\int$ is an integral sign;

C is a constant of integration.

NOTE:
Why the notation $\int f(x)\,dx$ for the general antiderivative $F(x) + C$??

see definite integrals...

Examples

1. $\int x\,dx = \frac{1}{2}x^2 + C$ since $\frac{d}{dx}\left(\frac{1}{2}x^2\right) = x$.

2. $\int \sin(4x)\,dx = -\frac{1}{4}\cos(4x) + C$ since $\frac{d}{dx}\left(-\frac{1}{4}\cos(4x)\right) = \sin(4x)$.

3. $\int \sec^2 x\,dx =??$

Linearity of Indefinite Integral

$$\int \big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx \;;$$

$$\int k\,f(x)\,dx = k \int f(x)\,dx, \quad \text{where } k \text{ is a } \underline{\text{constant}}.$$

NOTE:

$$\int f(x)\,g(x)\,dx \neq \int f(x)\,dx \cdot \int g(x)\,dx.$$

Example:

$$\int \big(3x + \sin(4x)\big)\,dx = \frac{3}{2}x^2 - \frac{1}{4}\cos(4x) + C.$$

§1.3 Indefinite Integrals of Elementary Functions

Every differentiation formula has an analogous integration formula.

E.g.

$\frac{d}{dx}\left(x^{r+1}\right) = (r+1)\, x^r$ implies $\boxed{\int x^r\, dx = \frac{1}{r+1}\, x^{r+1} + C \ \ \text{(where } r \neq -1\text{)}}$

Example:

$$\begin{aligned}\int \left(3\sqrt{x} + \frac{1}{x^2}\right) dx &= 3\int x^{1/2}\, dx + \int x^{-2}\, dx \\ &= 3\left(\frac{1}{1+\frac{1}{2}} x^{3/2}\right) + \frac{x^{-1}}{-2+1} + C \\ &= 2\, x\sqrt{x} - \frac{1}{x} + C.\end{aligned}$$

Furthermore, $(\sin x)' = \cos x$ implies $\boxed{\int \cos x \, dx = \sin x + C}$.

Similarly, $\boxed{\int \sin x \, dx = -\cos x + C}$ $\boxed{\int \sec^2 x \, dx = \tan x + C}$ ...

More generally, if $k \neq 0$, then

$$\boxed{\int \cos(kx) \, dx = \frac{1}{k} \sin(kx) + C} \quad \boxed{\int \sin(kx) \, dx = -\frac{1}{k} \cos(kx) + C} \quad \ldots$$

Example:

$$\begin{aligned} \int \cos^2(3x) \, dx &= \int \tfrac{1}{2}\big(1 + \cos(6x)\big) \, dx \\ &= \tfrac{1}{2} \int 1 \, dx + \tfrac{1}{2} \int \cos(6x) \, dx \\ &= \tfrac{1}{2} x + \tfrac{1}{2}\big(\tfrac{1}{6} \sin(6x)\big) + C \\ &= \tfrac{1}{2} x + \tfrac{1}{12} \sin(6x) + C. \end{aligned}$$

Integrals involving e^x:

Since $\frac{d}{dx}(e^x) = e^x$, so we have $\boxed{\int e^x \, dx = e^x + C}$.

Similarly, $\boxed{\int e^{kx} \, dx = \frac{1}{k} e^{kx} + C \quad (\text{where } k \neq 0)}$.

Related results:

$$\boxed{\int a^x \, dx} = \int e^{x \ln a} \, dx$$

$$= \frac{1}{\ln a} e^{x \ln a} + C$$

$$\boxed{= \frac{1}{\ln a} a^x + C, \quad \text{where } a > 0, \ a \neq 1}.$$

Further related results:

$$\boxed{\int \cosh x \, dx} = \int \tfrac{1}{2}\left(e^x + e^{-x}\right) dx$$

$$= \tfrac{1}{2}\left(e^x + \frac{e^{-x}}{-1}\right) + C = \tfrac{1}{2}\left(e^x - e^{-x}\right) + C$$

$$\boxed{= \sinh x + C}\,.$$

Similarly, $\boxed{\int \sinh x \, dx = \cosh x + C}$.

Integral leading to $\ln|x|$:

What is $\int \frac{1}{x}\,dx = \int x^{-1} dx$?? (Note: We cannot use the power rule!)

For $\underline{x > 0}$: we have $|x| = x$ so $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$.

For $\underline{x < 0}$: we have $|x| = -x$ so

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-x)' = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Combining the above observations, for all $x \neq 0$: $\frac{d}{dx}\ln|x| = \frac{1}{x}$ so

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C \quad \text{(where } x \neq 0\text{)}}.$$

Example:

1. $$\int \frac{x^4+1}{x}\,dx = \int \left(x^3+\frac{1}{x}\right)dx = \frac{1}{4}x^4 + \ln|x| + C.$$

2. $$\int \frac{x^4-2x^3+5x}{x^3}\,dx = ??$$

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n$$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Also $\displaystyle\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

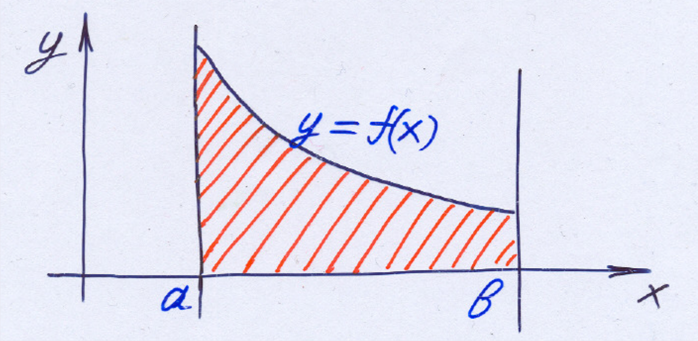
Proof: by induction...

Also $\displaystyle\sum_{i=1}^{n} r^{i-1} = 1 + r + r^2 + r^3 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}$ (where $r \neq 1$)

§2.1 Basic Area Problem

Find the area A bounded by

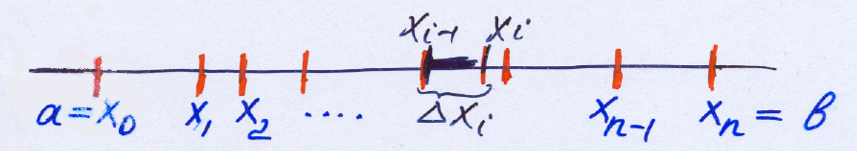
$$y = f(x) \geqslant 0, \ y = 0, \ x = a \text{ and } x = b \text{ (where } a < b\text{):}$$



Sollution:

- Introduce a partition $P = \{ x_0, x_1, x_2, \cdots, x_n \}$

  where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$:

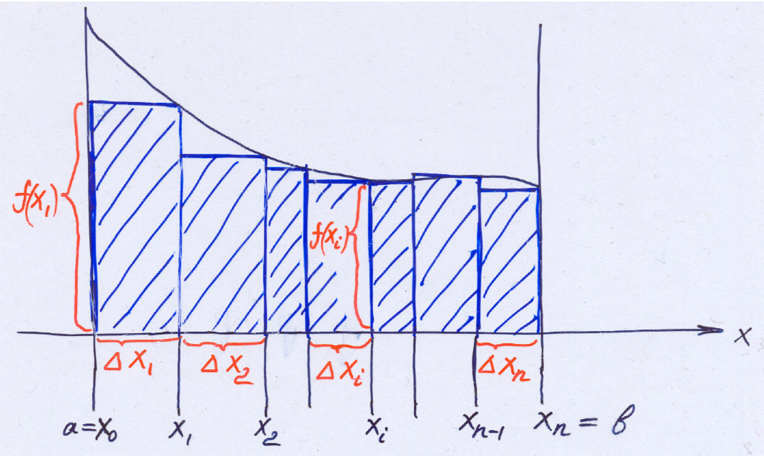


Here:

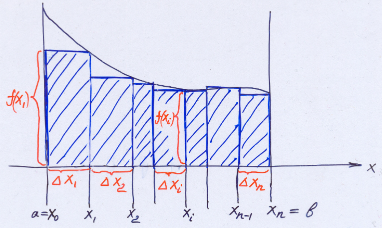
$[x_{i-1}, x_i]$ is the *i*-th subinterval,

$\triangle x_i = x_i - x_{i-1}$ is the *i*-th width.

For each $i = 1, 2, \cdots, n$,
construct a rectangle with base $\triangle x_i$ and height $f(x_i)$:



- For each $i = 1, 2, \cdots, n$,
  construct a rectangle with base $\triangle x_i$ and height $f(x_i)$:



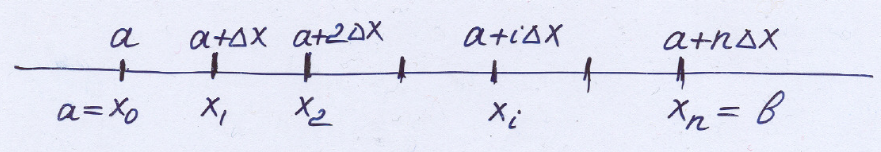
The sum of the areas of the rectangles is

$$S_n = f(x_1)\cdot\triangle x_1 + f(x_2)\cdot\triangle x_2 + \cdots + f(x_n)\cdot\triangle x_n = \sum_{i=1}^{n} f(x_i)\cdot\triangle x_i$$

is called a (right) Riemann Sum for $f(x)$ on $[a, b]$.

- As $n \to \infty$ and $\triangle x_i \to 0$, the rectangles get thinner so
  $S_n \to A =$ (the true area), or $\lim_{n\to\infty} S_n = A$.

- NOTE one particular case: if all rectangles have equal width $\triangle x$



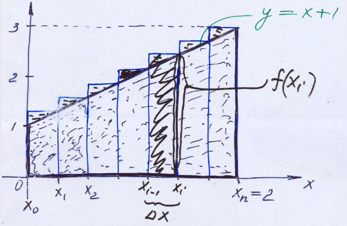
Then $\boxed{x_i = a + i \cdot \triangle x}$,

so $b = a + n \cdot \triangle x$ and $\boxed{\triangle x = \dfrac{b-a}{n}}$.

§2.2 Examples

Example A

Find the area of the trapezoid formed by
$y = x + 1, \ \ y = 0,$
$x = 0, \ \ x = 2$:



- Divide $[0, 2]$ into n equally spaced subintervals of width $\boxed{\triangle x = \frac{2-0}{n} = \frac{2}{n}}$.

- The partition is $x_i = 0 + i \cdot \triangle x$ so $\boxed{x_i = \frac{2i}{n}, \ \ i = 0, 1, \cdots, n}$.

  So $f(x_i) = 1 + x_i = 1 + \frac{2i}{n}$.

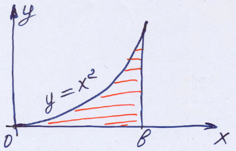
- Each rectangle has area: $f(x_i) \cdot \triangle x_i = \left(1 + \frac{2i}{n}\right) \cdot \left(\frac{2}{n}\right)$.
- Total area of all n rectangles:

$$
\begin{aligned}
S_n &= \left[\sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)\right] \cdot \left(\frac{2}{n}\right) \\
&= \left[\sum_{i=1}^{n} 1 + \frac{2}{n} \sum_{i=1}^{n} i\right] \cdot \frac{2}{n} \\
&= \left[n + \frac{2}{n} \cdot \frac{n(n+1)}{2}\right] \cdot \frac{2}{n} \\
&= [2n+1]\frac{2}{n} = 2\left[2 + \frac{1}{n}\right] = 4 + \frac{2}{n}.
\end{aligned}
$$

Now, we have $A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(4 + \frac{2}{n}\right) = 4.$ □

Example B

Find the area of the region bounded by
$y = x^2, \quad y = 0,$
$x = 0, \quad x = b$ (with $b > 0$):



- Divide $[0, b]$ into n equal subintervals of width $\boxed{\triangle x = \frac{b-0}{n} = \frac{b}{n}}$.
- The partition is $x_i = 0 + i \cdot \triangle x = \frac{ib}{n}, \ \ i = 0, 1, \cdots, n$.

  So $\quad f(x_i) = x_i^2 = \frac{i^2 b^2}{n^2}$.
- Total area of all n rectangles:

$$S_n = \sum_{i=1}^{n} f(x_i) \cdot \triangle x_i = \sum_{i=1}^{n} \left(\frac{i^2 b^2}{n^2} \right) \cdot \left(\frac{b}{n} \right) = \frac{b^3}{n^3} \sum_{i=1}^{n} i^2$$
$$= \frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{b^3}{6} \cdot 1\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right).$$

Now, we have $\quad A = \lim_{n\to\infty} S_n = \frac{b^3}{6} \cdot 1(1+0)(2+0) = \frac{b^3}{3}$. $\square$

CONCLUSION

$$\text{Area} = \lim_{n\to\infty} \underbrace{S_n}_{\text{Riemann Sum}}$$

Preliminary **Informal Definition of the Definite Integral**

$$\lim_{n\to\infty} \underbrace{S_n}_{\text{Riemann Sum}}$$

– is called the definite integral of $f(x)$ over (a, b);

– is denoted $\int_a^b f(x)\,dx$ $\Rightarrow$ $\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{S_n}_{\text{Riemann Sum}}}$

(t.b. discussed in Lecture 4...)

COMPARISON implies:

$$\boxed{\text{Area} = \lim_{n\to\infty} \underbrace{S_n}_{\text{Riemann Sum}} = \int_a^b f(x)\,dx}$$

Example C [useful for the Mid-Term!]

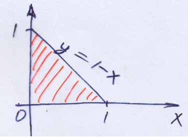
Interpret $S_n = \sum_{i=1}^{n} \left(\frac{n-i}{n^2} \right)$ as a sum of areas of rectangles, and hence evaluate $L = \lim_{n\to\infty} S_n$.

- Note that $\frac{n-i}{n^2} = \frac{n-i}{n} \cdot \frac{1}{n} = \left(1 - \frac{i}{n}\right) \cdot \frac{1}{n}$.
- Imagine n rectangles, each of width $\triangle x = \frac{1}{n}$ on $[0, 1]$.
- Then $x_i = 0 + i \cdot \triangle x = \frac{i}{n}$.
- So $\frac{n-i}{n^2} = (1 - x_i) \cdot \triangle x = f(x_i) \cdot \triangle x$ if <u>we choose</u> $f(x) = 1 - x$.
- So $L = \lim_{n\to\infty} \left[\sum_{i=1}^{n} \left(\frac{n-i}{n^2} \right) \right] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \triangle x$

  = area under the curve $y = 1 - x$ on $[0, 1]$:

  $\Rightarrow \boxed{L = \frac{1}{2}(1)(1) = \frac{1}{2}}$

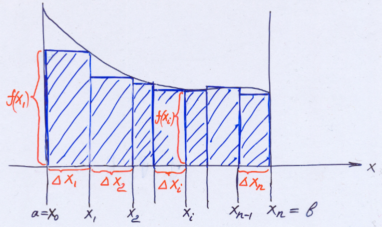
  (= area of a triangle with base $= 1$ and height $= 1$) $\square$



Lecture 3: §3.1 Partitions and Riemann Sums

- Recall a partition $P = \{ x_0, x_1, x_2, \cdots, x_n \}$ with $\triangle x_i = x_i - x_{i-1}$.
- Define $\boxed{\triangle x_{\max} = \max_{i=1,2,\cdots,n} \triangle x_i}$ = the width of the widest interval.

- *Lecture 2:* For each $i = 1, 2, \cdots, n$, construct a rectangle with base $\triangle x_i$ and height $f(x_i)$:



The sum of the areas of the rectangles is a right Riemann Sum

$$\boxed{S_n = f(x_1) \cdot \triangle x_1 + f(x_2) \cdot \triangle x_2 + \cdots + f(x_n) \cdot \triangle x_n = \sum_{i=1}^{n} f(x_i) \cdot \triangle x_i}$$

CONCLUSION from Lecture 2

$$\text{Area} = \lim_{n\to\infty} \underbrace{S_n}_{\text{Riemann Sum}}$$

"Practical" Informal Definition of the Definite Integral

$$\lim_{n\to\infty} \underbrace{S_n}_{\text{Riemann Sum}}$$

– is called the definite integral of $f(x)$ over (a, b);

– is denoted $\int_a^b f(x)\,dx$ $\Rightarrow$ $\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{S_n}_{\text{Riemann Sum}}}$

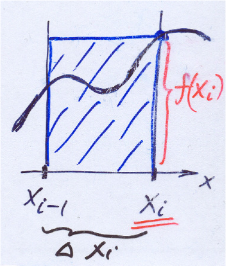
(t.b. discussed in Lecture 4...)

COMPARISON implies:

$$\boxed{\text{Area} = \lim_{n\to\infty} \underbrace{S_n}_{\text{Riemann Sum}} = \int_a^b f(x)\,dx}$$

- *Lecture 2:* recall that we used

right Riemann sums: $S_n = \sum_{i=1}^{n} f(x_i) \cdot \triangle x_i$

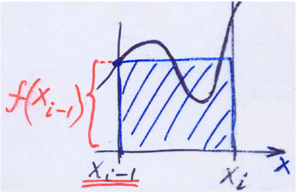


(called right since $f(x)$ is evaluated at the right-most point x_i in $[x_{i-1}, x_i]$)

- Similarly (left)$S_n = \sum_{i=1}^{n} f(x_{i-1}) \cdot \triangle x_i$

is called a left Riemann sum

(since $f(x)$ is evaluated at the left-most point x_{i-1} in $[x_{i-1}, x_i]$)

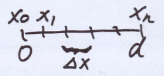


Example A:

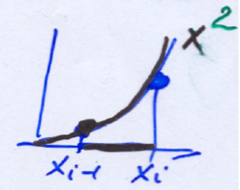
(i) Evaluate (left)S_n and (right)S_n for $f(x) = x^2$ on $[0, d]$ (where $d > 0$), with partition P of n equally spaced subintervals.

(ii) Hence check that $\lim_{n\to\infty} \text{(left)}S_n = \lim_{n\to\infty} \text{(right)}S_n = \text{Area}$.

Solution: We have $\triangle x = \frac{d}{n}$, $x_i = \frac{id}{n}$ from



Note:
$f(x_{i-1}) \cdot \triangle x = (x_{i-1})^2 \cdot \frac{d}{n}$



$$\Rightarrow \text{(left)}S_n = \sum_{i=1}^{n} f(x_{i-1}) \cdot \triangle x = \sum_{i=1}^{n} (x_{i-1})^2 \cdot \frac{d}{n}$$

$$\Rightarrow \text{(left)}S_n = \sum_{i=1}^{n} f(x_{i-1}) \cdot \triangle x = \sum_{i=1}^{n} (x_{i-1})^2 \cdot \tfrac{d}{n}$$

$$= \sum_{i=1}^{n} \left[(i-1)^2 \tfrac{d^2}{n^2}\right] \cdot \tfrac{d}{n} = \tfrac{d^3}{n^3}\left(0^2 + 1^2 + \cdots + (n-1)^2\right) = \tfrac{d^3}{n^3} \sum_{j=1}^{n-1} j^2$$

$$= \tfrac{d^3}{n^3} \tfrac{[n-1]([n-1]+1)(2[n-1]+1)}{6} = \tfrac{d^3}{n^3} \tfrac{[n-1](n)(2n-1)}{6} = \tfrac{d^3}{3}(1 - \tfrac{1}{n})(1 - \tfrac{1}{2n}).$$

$$\boxed{\Rightarrow \lim_{n\to\infty} \text{(left)}S_n = \lim_{n\to\infty} \tfrac{d^3}{3}(1 - \tfrac{1}{n})(1 - \tfrac{1}{2n}) = \tfrac{d^3}{3}}$$

Similarly (see Lecture 2, p. 9): $\text{(right)}S_n =$

$$\sum_{i=1}^{n} f(x_i) \cdot \triangle x = \sum_{i=1}^{n} \left(\tfrac{i^2 d^2}{n^2}\right) \cdot \tfrac{d}{n} = \tfrac{d^3}{3}(1 + \tfrac{1}{n})(1 + \tfrac{1}{2n}).$$

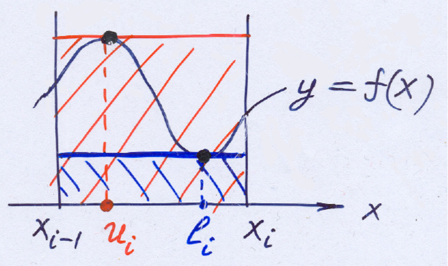
$$\boxed{\Rightarrow \lim_{n\to\infty} \text{(right)}S_n = \lim_{n\to\infty} \tfrac{d^3}{3}(1 + \tfrac{1}{n})(1 + \tfrac{1}{2n}) = \tfrac{d^3}{3}}$$

Finally, $\text{Area} = \lim_{n\to\infty} \text{(left)}S_n = \lim_{n\to\infty} \text{(right)}S_n = \tfrac{d^3}{3}$; we are DONE.

- Lower/Upper Riemann Sums:

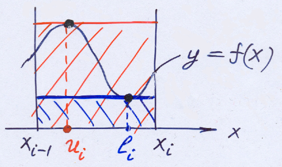
Let $f(x)$ be continuous on $[x_{i-1}, x_i]$.Then by the Max/Min Theorem,

$\exists\ l_i,\ u_i \in [x_{i-1}, x_i]$ such that $f(l_i) \leqslant f(x) \leqslant f(u_i)$ for all $x \in [x_{i-1}, x_i]$



$$\underbrace{f(l_i)\cdot\triangle x_i}_{\text{area of the lowest rectangle}} \leqslant f(x)\cdot\triangle x_i \leqslant \underbrace{f(u_i)\cdot\triangle x_i}_{\text{area of the upper-most rectangle}}$$

- Lower/Upper Riemann Sums (continued):



Definition

The Lower Riemann Sum for $f(x)$ and a partition P is

$$L_n(f,P) = f(l_1)\cdot\triangle x_1 + f(l_2)\cdot\triangle x_2 + \cdots + f(l_n)\cdot\triangle x_n = \sum_{i=1}^{n} f(l_i)\cdot\triangle x_i$$

The Upper Riemann Sum for $f(x)$ and a partition P is

$$U_n(f,P) = f(u_1)\cdot\triangle x_1 + f(u_2)\cdot\triangle x_2 + \cdots + f(u_n)\cdot\triangle x_n = \sum_{i=1}^{n} f(u_i)\cdot\triangle x_i$$

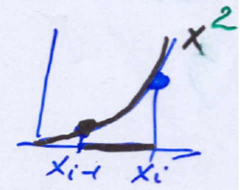
Example A' (related to Example A):

(i) Evaluate $L_n(f, P)$ and $U_n(f, P)$ for $f(x) = x^2$ on $[0, d]$ (where $d > 0$), with partition P of n equally spaced subintervals.

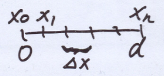
(ii) Hence check that $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = \text{Area}$.

Solution:

Note: $f(x) = x^2$ is increasing on each $[x_{i-1}, x_i]$, so $l_i = x_{i-1}$, $u_i = x_i$



where



$$\Rightarrow L_n = \sum_{i=1}^{n} f(l_i) \cdot \triangle x = \sum_{i=1}^{n} f(x_{i-1}) \cdot \triangle x = (\text{left})S_n$$

$$\Rightarrow U_n = \sum_{i=1}^{n} f(u_i) \cdot \triangle x = \sum_{i=1}^{n} f(x_i) \cdot \triangle x = (\text{right})S_n$$

Using Example A:

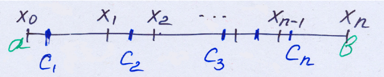
$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} (\text{left}) S_n = \lim_{n\to\infty} \frac{d^3}{3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right) = \frac{d^3}{3}$$

and

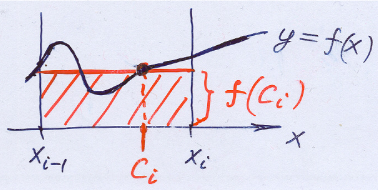
$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} (\text{right}) S_n = \lim_{n\to\infty} \frac{d^3}{3}\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right) = \frac{d^3}{3}$$

Finally, $\text{Area} = \lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = \frac{d^3}{3}$; we are DONE.

- General Riemann Sums:
Choose a point c_i in $[x_{i-1}, x_i]$ randomly,and construct the rectangle with base $\triangle x_i$, height $f(c_i)$, and area $f(c_i) \cdot \triangle x_i$:



$\Rightarrow$



Definition

The General Riemann Sum for $f(x)$ and any partition P:

$$R_n(f, P, c) = f(c_1) \cdot \triangle x_1 + f(c_2) \cdot \triangle x_2 \cdots + f(c_n) \cdot \triangle x_n = \sum_{i=1}^{n} f(c_i) \cdot \triangle x_i$$

Three further Remarks:

1. Lower/Upper Riemann sums are particular cases of general Riemann sums $R_n(f, P, c)$ with $c_i = l_i$ and $c_i = u_i$, respectively.

   ....

2. Left/Right Riemann sums are particular cases of general Riemann sums $R_n(f, P, c)$ with $c_i = x_{i-1}$ and $c_i = x_i$, respectively.

   ....

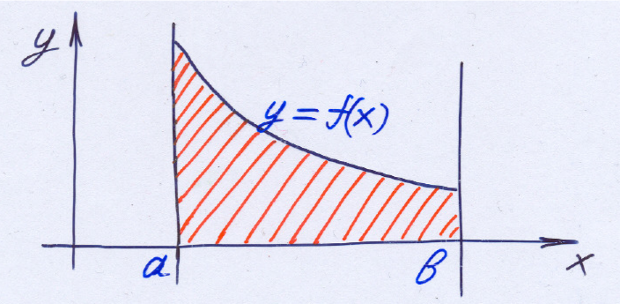
3. All Riemann sums lie between $L_n(f, P)$ and $U_n(f, P)$:

   $$\boxed{L_n(f, P) \leqslant R_n(f, P, c) \leqslant U_n(f, P)}$$

   since $f(l_i) \cdot \triangle x_i \leqslant f(c_i) \cdot \triangle x_i \leqslant f(u_i) \cdot \triangle x_i$

   (the asserted remark is obtained by an application of $\sum_{i=1}^{n}$ to the above double inequality...)

   ....



—For this area problem with $f(x) \geqslant 0$, we have

$$\text{Area} = \lim_{n\to\infty} R_n(f, P, c) \text{ for ANY type of Riemann sums!}$$

"Practical" **Informal Definition of the Definite Integral**

$$\lim_{n\to\infty} \underbrace{R_n}_{\text{ANY Riemann Sum}} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\cdot\triangle x_i$$

(provided this *lim* is the same for any Riemann Sum...)

– is called the definite integral of $f(x)$ on (a, b);

– is denoted $\int_a^b f(x)\,dx \quad \Rightarrow \quad \boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{R_n}_{\text{ANY Riemann Sum}}}$

Here the notation $\int_a^b f(x)\,dx$ replaces $\sum_{i=1}^{n} f(c_i)\triangle x_i$:

$\int$ is an integral sign (replaces $\sum$);

a, b are limits of integration (replace $i = 1$ and $i = n$);

dx is the differential of x (replaces $\triangle x_i$).

Compare:

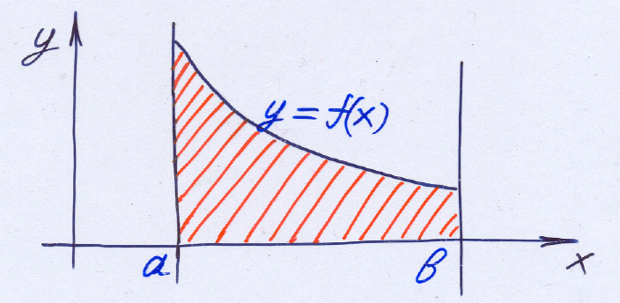
- Definite Integral: $\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\cdot\triangle x_i}$ is a number;

  here x is a dummy variable: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.

- Indefinite Integral: $\boxed{\int f(x)\,dx = F(x) + C}$ is a function of x.

COMPARISON implies:

$$\text{Area} = \lim_{n\to\infty} \underbrace{S_n}_{\text{ANY Riemann Sum}} = \int_a^b f(x)\,dx \quad \Leftrightarrow \quad f(x) \geqslant 0$$

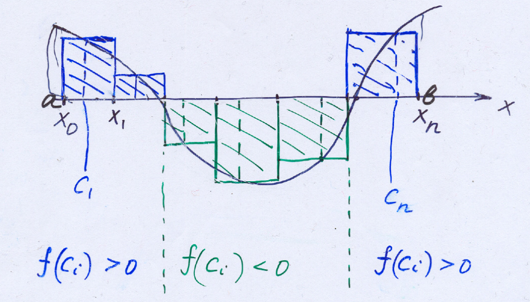


NOTE: If $f(x) \geqslant 0$ is NOT satisfied, the above is NOT TRUE

(see next page...)

Remark: if $f(x)$ changes sign, then

$$R_n(f,P,c) = \sum_{i=1}^{n} \underbrace{f(c_i)}_{\text{may be } < 0} \cdot \triangle x_i$$ is a sum of signed areas of rectangles:



§3.2* The Definite Integral — FORMAL Definition

Formal Definition, which works for ALL functions:

Definition: Definite Integral

Suppose there is a unique number I such that for every partition P of $[a, b]$ we have $L_n(f, P) \leqslant I \leqslant U_n(f, P)$.

Then we say that f is integrable on $[a, b]$, and call I the definite integral of f on $[a, b]$: $I = \int_a^b f(x)\, dx$.

NOTE: —If the function f is integrable,
then Formal and Informal Definitions give the same answer

(this is discussed in §3.3).

—The Informal Definition does NOT work, if the function f is NOT integrable

(see the final example of this Lecture 3).

Example A**: Show that $f = x^2$ is integrable on $[0, d]$ (where $d > 0$) and evaluate $\int_0^d x^2\,dx$.

Solution: Recall that in §3.1 for this function we obtained

$$L_n(f,P) = \tfrac{d^3}{3}(1-\tfrac{1}{n})(1-\tfrac{1}{2n})$$
$$U_n(f,P) = \tfrac{d^3}{3}(1+\tfrac{1}{n})(1+\tfrac{1}{2n})$$

Now, $L_n(f,P) \leqslant \frac{d^3}{3} \leqslant U_n(f,P)$ for all n;

and $\lim_{n\to\infty} L_n(f,P) = \frac{d^3}{3} = \lim_{n\to\infty} U_n(f,P)$.

Consequently,

$I = \frac{d^3}{3}$ is the unique number between all $L_n(f,P)$ and all $U_n(f,P)$,

so x^2 is integrable and $\int_0^d x^2\,dx = \frac{d^3}{3}$. □

§3.3 Informal Definition works for Integrable Functions

Proof***: A general Riemann sum $R_n(f, P, c) = \sum_{i=1}^{n} f(c_i) \cdot \triangle x_i$ satisfies

$$L_n(f, P) \leqslant R_n(f, P, c) \leqslant U_n(f, P) \quad \text{(see §3.1).}$$

- For $f(x)$ integrable on $[a, b]$:

  let $n \to \infty$,and also let $\triangle x_{\max} = \max_{i=1,2,\cdots,n} \triangle x_i \to 0$.

  Then $L_n(f, P)$ and $U_n(f, P)$ converge to each other and to the integral I:

$$\lim_{n\to\infty} L_n(f, P) = I = \lim_{n\to\infty} U_n(f, P).$$

  Hence, by the Squeeze Theorem,

$$\lim_{n\to\infty} R_n(f, P, c) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \cdot \triangle x_i = \int_a^b f(x)\, dx$$

(as $R_n(f, P, c) = \sum_{i=1}^{n} f(c_i) \cdot \triangle x_i$ remains between $L_n(f, P)$ and $U_n(f, P)$).

- Some good news: there are quite many integrable functions...

Theorem

If $f(x)$ is continuous on $[a, b]$, then $f(x)$ is integrable on $[a, b]$.

- Example: [Mid-Term] Write as a definite integral the quantity

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(1+\frac{2i}{n}\right)^{1/3}.$$

Solution: Let $\triangle x = \frac{1}{n}$, $[a,b]=[0,1]$, $x_i = \frac{i}{n}$.

Then $L = \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{2i}{n}\right)^{1/3} \cdot \triangle x}_{\text{Right R. Sum}} = \int_0^1 (1+2x)^{1/3}dx.$ □

NOTE: At the Mid-Term, you'll have to also evaluate the resulting integral using any technique of integration that we learn by then....

Example***: a function NOT integrable on $[0, 1]$:

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ & \text{(i.e., } x \text{ is a terminating or repeating decimal,} \\ & \text{such as } 0.9 \text{ or } 0.\overline{31}) \\ 0, & \text{if } x \text{ is irrational,} \\ & \text{(e.g., } x = \frac{1}{\sqrt{2}}, \frac{1}{\pi} \text{...)} \end{cases}$$

Note that for this function:

$f(0.1) = f(0.\bar{3}) = f(0.9) = 1$, while $f(\frac{1}{\sqrt{2}}) = f(\frac{1}{\pi}) = 0$.

$\Rightarrow$ for any partition P of $[0, 1]$, each $[x_{i-1}, x_i]$ includes rational and irrational points, $\Rightarrow$ $\boxed{f(l_i) = 0}$ and $\boxed{f(u_i) = 1}$.

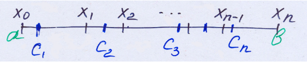
$$L_n(f, P) = \sum_{i=1}^{n} \underbrace{f(l_i)}_{=0} \cdot \triangle x_i = 0, \quad U_N(f, P) = \sum_{i=1}^{n} \underbrace{f(u_i)}_{=1} \cdot \triangle x_i = \sum_{i=1}^{n} \triangle x_i = 1.$$

$\Rightarrow$ between $L_n(f, P) = 0$ and $U_n(f, P) = 1$ there are many numbers, e.g., 0.5, 0.7,... — i.e. the number is not unique.

$\Rightarrow$ this function f is NOT integrable. $\square$

Recall (from §3.3): if $f(x)$ is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \underbrace{\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\cdot\triangle x_i}_{\text{T.b. discussed in more detail!}}$$

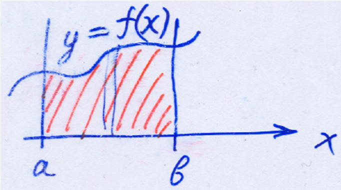


(Whenever $n\to\infty$, let $\max\limits_{i=1,2,\cdots,n} \triangle x_i \to 0$.)

Case (i) If $f(x) \geqslant 0$ for all $x \in [a, b]$, then (Lecture 2)

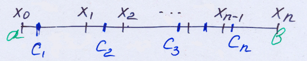
$$\boxed{\int_a^b f(x)\,dx = A} = \text{the area}$$

bounded by $y = f(x)$, $y = 0$, $x = a$, $x = b$.



Recall (from §3.3): if $f(x)$ is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\cdot\triangle x_i}_{\text{T.b. discussed in more detail!}}$$



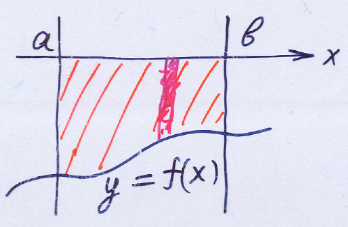
(Whenever $n \to \infty$, let $\max\limits_{i=1,2,\cdots,n} \triangle x_i \to 0$.)

Case (ii) If $f(x) \leqslant 0$ for all $x \in [a, b]$,

then $f(c_i)\cdot\triangle x_i \leqslant 0$,

and $\boxed{\int_a^b f(x)\,dx = -A}$,

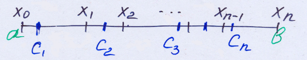
where A is the area:



Recall (from §3.3): if $f(x)$ is integrable on $[a, b]$, then

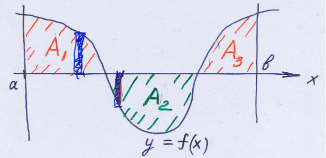
$$\int_a^b f(x)\,dx = \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\cdot \triangle x_i}$$

T.b. discussed in more detail!



(Whenever $n \to \infty$, let $\max\limits_{i=1,2,\cdots,n} \triangle x_i \to 0$.)

Case (iii) If $f(x)$ changes sign:

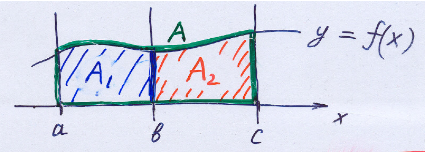


Recall that $\sum_{i=1}^{n} f(c_i)\cdot \triangle x_i$ is the sum of signed areas of rectangles.

Hence $\boxed{\int_a^b f(x)\,dx = A_1 - A_2 + A_3}$

§4.1.1 Additivity

$$\underbrace{\int_a^b f(x)\,dx}_{A_1} + \underbrace{\int_b^c f(x)\,dx}_{A_2} = \underbrace{\int_a^c f(x)\,dx}_{A}$$



Particular Cases:

- Set $c = b$. Then $\boxed{\int_b^b f(x)\,dx = 0}$.
- Set $c = a$. Then $\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0$.

  So $\boxed{\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx}$.

§4.1.2 Linearity

$$\int_a^b \big(A\,f(x) + B\,g(x)\big)\,dx = A\int_a^b f(x)\,dx + B\int_a^b g(x)\,dx$$

§4.1.3 Inequalities

- If $f(x) \leqslant g(x)$ for all $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx$ (here $a \leqslant b$).

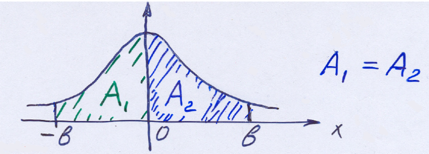
- Triangle Inequality

$$\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx$$ (where $a \leqslant b$).

This follows from $\displaystyle\left|\sum_{i=1}^{n} f(c_i)\cdot\triangle x_i\right| \leqslant \sum_{i=1}^{n}\big|f(c_i)\big|\cdot\triangle x_i$.

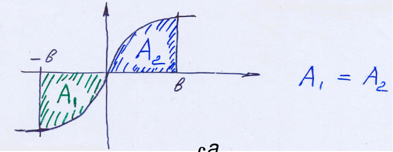
§4.1.4 Even/Odd Functions

- $f(x)$ is even on $[-b, b]$ if $\boxed{f(-x) = f(x)}$ for all $x \in [-b, b]$.



$$\boxed{\int_{-b}^{b} f(x)\,dx =} A_1 + A_2 = 2A_2 = \boxed{= 2\int_0^b f(x)\,dx}$$

- $f(x)$ is odd on $[-b, b]$ if $\boxed{f(-x) = -f(x)}$ for all $x \in [-b, b]$.



$$\boxed{\int_{-b}^{b} f(x)\,dx =} -A_1 + A_2 \boxed{= 0}$$

- <u>Examples:</u> $\displaystyle\int_{-a}^{a} \sin(kx)\,dx = 0;$

$$\int_{-5}^{5} x^2 \sin x\,dx = 0; \qquad \int_{-\pi}^{\pi} x^{2n+1}\,dx = 0 \ (n \geqslant 0 \text{ is integer}).$$

§4.2 Average Value of a Function

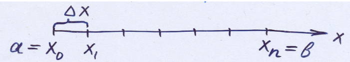
Definition

If $f(x)$ is integrable on $[a, b]$, then
the average value or mean value of f on $[a, b]$ is

$$\boxed{\bar{f} = \frac{1}{b-a} \int_a^b f(x)\,dx}.$$

Hint (motivation for this definition):

Define the partition:



with $x_i = a + i \cdot \triangle x$ and $\triangle x = \frac{b-a}{n}$.

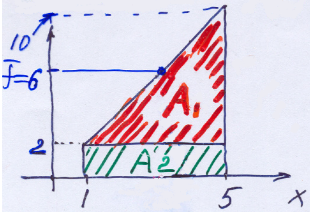
Then one can interpret $\bar{f}$ as

$$\bar{f} = \lim_{n\to\infty} \frac{f(x_0) + f(x_1) + \cdots + f(x_{n-1})}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_{i-1})$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1}) \frac{\triangle x}{b-a} = \frac{1}{b-a} \int_a^b f(x)\, dx,$$

since $\sum_{i=1}^{n} f(x_{i-1}) \cdot \triangle x$ is a left Riemann sum. □

Example: Consider $f(x) = 2x$ on $[1, 5]$.



The average value $\bar{f} = \frac{1}{5-1}\int_1^5 2x\,dx = \frac{1}{4}(A_1 + A_2)$.

Triangle: $A_1 = \frac{1}{2} \cdot 4 \cdot 8 = 16$;

Rectangle: $A_2 = 4 \cdot 2 = 8$.

$\Rightarrow \quad \bar{f} = \frac{1}{4}(16 + 8) = 6$ $\square$

Mean-Value Theorem for Integrals

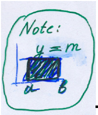
If $f(x)$ is continuous on $[a, b]$, then $\exists\ c \in [a, b]$ such that $\boxed{f(c) = \bar{f}}$.

Example: Driving from Limerick to Cork, at some time during the trip you are traveling at your average speed.

Theorem Proof: By the Max/Min Theorem,

$$\min_{x\in[a,b]} f(x) = m \leqslant f(x) \leqslant M = \max_{x\in[a,b]} f(x).$$

Hence, $\underbrace{\int_a^b m\, dx}_{=m(b-a)} \leqslant \int_a^b f(x)\, dx \leqslant \underbrace{\int_a^b M\, dx}_{=M(b-a)}$



—was used here.

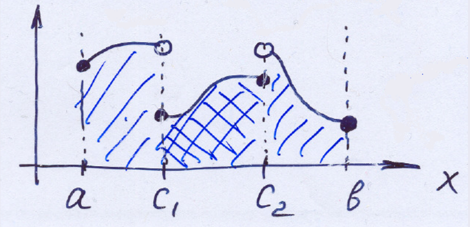
Now, $m \leqslant \bar{f} \leqslant M$.

Finally, by the Intermediate Value Theorem, there exists $c \in [a, b]$ such that $f(c) = \bar{f}$. □

§4.3 Definite Integrals of Piecewise Continuous Functions

Definition

Let $a = c_0 < c_1 < \cdots < c_n = b$ be a set of points on $[a, b]$.
Suppose $f(x)$ is continuous on each open interval (c_{i-1}, c_i):

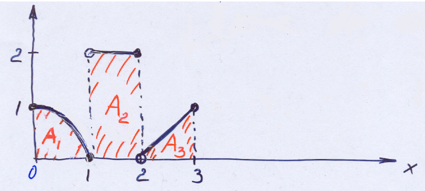


Then $f(x)$ is called piecewise continuous on $[a, b]$ and

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \cdots + \int_{c_{n-1}}^{b} f(x)\,dx$$

Example:

$$f(x) = \begin{cases} \sqrt{1-x^2} & \text{for} \quad 0 \leqslant x \leqslant 1, \\ 2 & \text{for} \quad 1 < x \leqslant 2, \\ x-2 & \text{for} \quad 2 < x \leqslant 3. \end{cases}$$



$$\begin{aligned} \int_0^3 f(x)\,dx &= \int_0^1 \sqrt{1-x^2}\,dx + \int_1^2 2\,dx + \int_2^3 (x-2)\,dx \\ &= A_1 + A_2 + A_3 \\ &= \tfrac{1}{4}\pi \cdot 1^2 + 2 \cdot 1 + \tfrac{1}{2} \cdot 1 \cdot 1 \\ &= \tfrac{\pi}{4} + \tfrac{5}{2}. \qquad \square \end{aligned}$$

Lecture 5 The Fundamental Theorem of Calculus:
§5.1 Statement of the Theorem

Theorem, Part I

Let $f(x)$ be continuous on an interval I containing the point a. Define $\boxed{F(x) = \int_a^x f(t)\,dt}$. Then $F(x)$ is differentiable on I and $\boxed{\frac{d}{dx}F(x) = f(x)}$.

Remark: Part I says that $F(x)$ is an antiderivative of $f(x)$.

Theorem, Part II

Let $f(x)$ be continuous on $I = [a, b]$, and $G(x)$ be any antiderivative of $f(x)$ (i.e. $G'(x) = f(x)$). Then $\boxed{\int_a^b f(x)\,dx = G(x)\Big|_a^b = G(b) - G(a)}$.

Notation: Here we used new notation:

$G(x)\Big|_a^b = G(b) - G(a)$; $F(x)\Big|_a^b = F(b) - F(a)$; e.g. $x^2\Big|_1^2 = 2^2 - 1^2 = 3$.

§5.2 Proof of the Fundamental Theorem of Calculus

Theorem, Part I

Let $f(x)$ be continuous on an interval I containing the point a. Define $\boxed{F(x) = \int_a^x f(t)\,dt}$. Then $F(x)$ is differentiable on I and $\boxed{\frac{d}{dx}F(x) = f(x)}$.

<u>Proof of Part I:</u> Recall that $F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$.

Here $F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt = h \cdot f(c)$

(by the Mean Value Theorem for integrals; §4.2, p. 10),

where $c \in [x, x+h]$ (and c depends on h).

Now, let $h \to 0$, then $F'(x) = \lim_{h \to 0} \frac{h \cdot f(c)}{h} = \lim_{h \to 0} f(c) = f(x)$. $\square$

Theorem, Part II

Let $f(x)$ be continuous on $I = [a, b]$, and $G(x)$ be any antiderivative of $f(x)$ (i.e. $G'(x) = f(x)$). Then $\boxed{\int_a^b f(x)\,dx = G(x)\Big|_a^b = G(b) - G(a)}$.

Proof of Part II:

Use $F(x) = \int_a^x f(t)\,dt$ from Part I, for which we already proved that $F(x)$ is a particular antiderivative of $f(x)$.

$$\Rightarrow \quad G(x) = F(x) + C$$

$$\Rightarrow \quad G(a) = F(a) + C = C, \qquad G(b) = F(b) + C.$$

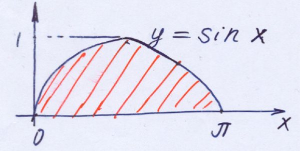
Now, $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = F(b) = G(b) - C = G(b) - G(a).$ □

§5.3 Examples for Part II

NOTE: Part II related Definite and Indefinite Integrals:

$$\underbrace{\int_a^b f(x)\,dx}_{\substack{\text{definite:}\\ \text{a number}}} = \Big(\underbrace{\int f(x)\,dx}_{\substack{\text{indefinite:}\\ \text{a function}}} \Big)\Big|_a^b .$$

Example 1: Find the area between $y = \sin x$ and $y = 0$ for $0 \leqslant x \leqslant \pi$.

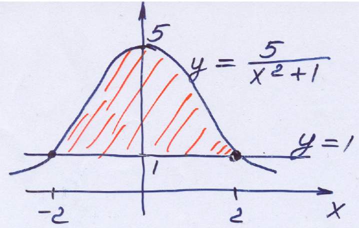


Solution:

$$A = \int_0^{\pi} \sin x\,dx = \Big(\underbrace{-\cos x}_{\substack{\text{antiderivative}\\ \text{of } \sin x}} \Big)\Big|_0^{\pi}$$

$$= (-\cos\pi) - (-\cos 0) = 1 - (-1) = 2. \quad \square$$

Example 2: Find the area bounded by $y = \frac{5}{x^2+1}$ and $y = 1$.



S: Intersections:

$$\frac{5}{x^2+1} = 1 \Leftrightarrow 5 = x^2 + 1 \Leftrightarrow x = \pm 2.$$

$$A = \int_{-2}^{2} (\text{Upper} - \text{Lower})\, dx = \int_{-2}^{2} \left(\frac{5}{x^2+1} - 1 \right) dx$$

$$= \Big(\underbrace{5 \tan^{-1} x}_{\substack{\text{antiderivative} \\ \text{of } \frac{5}{x^2+1}}} - \underbrace{x}_{\substack{\text{antiderivative} \\ \text{of } 1}} \Big) \Big|_{-2}^{2}$$

$$= \left(5 \tan^{-1} 2 - 2\right) - \left(5 \tan^{-1}(-2) - (-2)\right) = 10 \tan^{-1} 2 - 4 \approx 7.071. \; \square$$

Example 3: Find the average value of $y = e^{-x} + \cos x$ on $[0, \frac{\pi}{2}]$.

Solution:

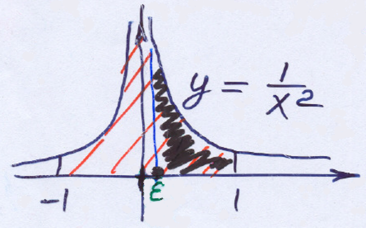
$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\frac{\pi}{2}-0}\int_0^{\frac{\pi}{2}} (e^{-x} + \cos x)\,dx$$

$$= \frac{2}{\pi}\Big(\underbrace{-e^{-x}}_{\text{antiderivative of } e^{-x}} + \underbrace{\sin x}_{\text{antiderivative of } \cos x} \Big)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left(\left(-e^{-\frac{\pi}{2}} + \sin\frac{\pi}{2}\right) - \left(-e^{-0} + \sin 0\right)\right)$$

$$= \frac{2}{\pi}\left(\left(-e^{-\pi/2} + 1\right) - \left(-1 + 0\right)\right) = \frac{2}{\pi}\left(2 - e^{-\pi/2}\right).$$

Example 4: Be careful when integrating near infinities!



Using Part II:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^{1} = -\frac{1}{1} + \frac{1}{-1} = -2$$

But the area has to be positive???

NOTE: $\frac{1}{x^2}$ is not continuous on $[-1, 1]$ as $\lim_{x\to 0} \frac{1}{x^2} = \infty$ so the above approach to evaluate the area doesn't give a correct answer...

INSTEAD, try $\int_{\varepsilon}^{1} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{\varepsilon}^{1} = -\frac{1}{1} + \frac{1}{\varepsilon} \to +\infty$ as $\varepsilon \to 0^+$.

So the area in this example is infinite!

EX: apply a similar approach to evaluate $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \ldots$

§5.4 Part I: Extensions and Examples

- Part I says: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

- Generalize for $y = \int_a^{u(x)} f(t)\,dt$:

Use the Chain Rule: let $y = y(u(x))$, then $\frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{=f(u)} \cdot \frac{du}{dx}$

as $y = y(u) = \int_a^u f(t)\,dt$ so indeed, by part I, $\frac{dy}{du} = f(u)$.

Now, $\boxed{\frac{dy}{dx} = f(u(x)) \cdot \frac{du}{dx}}$.

Example 1:

$$F(x) = \int_x^3 e^{-t^2}\,dt = \underbrace{-\int_3^x}_{\text{to apply Part I}} e^{-t^2}\,dt \quad \Rightarrow \quad \frac{d}{dx}F(x) = -e^{-x^2}. \qquad \square$$

Example 2: $G(x) = x^2 \cdot \int_{-4}^{5x} e^{-t^2}\,dt.$

By the Product Rule:

$$\frac{d}{dx}G(x) = 2x \cdot \int_{-4}^{5x} e^{-t^2}\,dt + x^2 \cdot \frac{d}{dx}\int_{-4}^{5x} e^{-t^2}\,dt$$

$$= 2x \cdot \int_{-4}^{5x} e^{-t^2}\,dt + x^2 \cdot e^{-(5x)^2}\,\frac{d}{dx}(5x)$$

$$= 2x \cdot \int_{-4}^{5x} e^{-t^2}\,dt + 5x^2 \cdot e^{-25x^2}. \qquad \square$$

Example 3:

$$H(x) = \int_{x^2}^{x^3} e^{-t^2}\,dt \underbrace{=}_{\text{Trick}} \int_{x^2}^{0} e^{-t^2}\,dt + \int_{0}^{x^3} e^{-t^2}\,dt$$

$$= -\int_0^{x^2} e^{-t^2}\,dt + \int_0^{x^3} e^{-t^2}\,dt$$

$$\frac{dH}{dx} = -e^{-(x^2)^2} \cdot (x^2)' + e^{-(x^3)^2} \cdot (x^3)' = -2x \cdot e^{-x^4} + 3x^2 \cdot e^{-x^6}. \quad \square$$

§5.5 Evaluation of Riemann Sums

Now we can evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \cdot \triangle x_i$ using antiderivatives!

Riemann Sum $\longrightarrow$ Definite Integral $\underbrace{\longrightarrow}_{\text{Part II}}$ Antiderivative

Example (Mid-Term): Evaluate $L = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \cos\left(\frac{j\pi}{2n}\right)$.

S: The sum involves $\cos x$ at the right endpoints of the partition:

$0, \ \frac{\pi}{2n}, \ 2\frac{\pi}{2n}, \ \cdots ; n\frac{\pi}{2n}$ of $[0, \frac{\pi}{2}]$.

For this partition, $\triangle x = \frac{\pi}{2n} \Rightarrow \frac{1}{n} = \frac{2\triangle x}{\pi}$, $x_j = \frac{j\pi}{2n}$.

$$\Rightarrow \ L = \lim_{n\to\infty} \frac{2\triangle x}{\pi} \sum_{j=1}^{n} \cos(x_j) = \frac{2}{\pi} \lim_{n\to\infty} \sum_{j=1}^{n} \cos(x_j) \cdot \triangle x$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \cos x \, dx = \frac{2}{\pi} \sin x \Big|_0^{\pi/2} = \frac{2}{\pi} \left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{2}{\pi}(1-0) = \frac{2}{\pi}.$$

Recall the Chain Rule: $\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big) \cdot g'(x)$.

This implies: $\boxed{\int f'\big(g(x)\big) \cdot g'(x)\, dx = \int \frac{d}{dx} f\big(g(x)\big)\, dx = f\big(g(x)\big) + C}$

To apply this method:

- Identify a good $u = g(x)$
- Construct $\boxed{\frac{du}{dx} \cdot dx = du}$
- Write the remainder of the integrand in terms u only (No x is allowed!)
- Now $\int f'(u)\, \frac{du}{dx} \cdot dx = \underbrace{\int f'(u)\, du}_{\text{NO } x \text{ here!}} = f(u) + C$
- Finally, use $u = g(x)$ to get the answer $f\big(g(x)\big) + C$ in terms of the original variable x. □

Examples:

1. $\int \frac{x}{x^2+1}\,dx$

NOTE: $x \cdot dx = \frac{1}{2}\,(x^2)'\,dx = \frac{1}{2}\,(x^2+1)'\,dx.$

S: Use $u = x^2 + 1$ with $du = 2x\,dx$

$\Rightarrow \quad x \cdot dx = \frac{1}{2} \underbrace{\frac{du}{dx}}_{=2x} dx = \frac{1}{2} du;$ while $\frac{1}{x^2+1} = \frac{1}{u}$.

Now, $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C. \quad \square$

Examples:

2. $\int \frac{\sin(3\ln x)}{x}\,dx$ (where $x > 0$)

Use $u = 3\ln x$, then $du = \underbrace{\frac{3}{x}}_{\frac{du}{dx}} dx$, so

$$\int \sin u \cdot \frac{du}{3} = \frac{1}{3}\left(-\cos u\right) + C = -\frac{1}{3}\cos(3\ln x) + C. \qquad \square$$

Examples:

3. $\int e^x \sqrt{1+e^x}\, dx$

S: Use $u = 1 + e^x$, then $du = \underbrace{e^x}_{(1+e^x)'} dx$, so

$$\int \sqrt{1+e^x}\,(e^x\,dx) = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\left(1+e^x\right)^{3/2} + C. \qquad \square$$

§6.2 Substitution in Definite Integrals

Similarly,

$$\boxed{\int_a^b f'\big(g(x)\big) \cdot g'(x)\,dx = f\big(g(x)\big)\Big|_a^b = f\big(g(b)\big) - f\big(g(a)\big) = f(B) - f(A)}$$

where $A = g(a)$ and $B = g(b)$.

Example 1: $I = \int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,dx$.

Method I: First do the Indefinite Integral.

Note: $I = \int_0^8 \cos\sqrt{x+1} \cdot \frac{dx}{\sqrt{x+1}}$,

so use $u = \sqrt{x+1}$ with $du = \left(\sqrt{x+1}\right)' dx = \frac{1}{2}\frac{1}{\sqrt{x+1}}\,dx$.

We now have: $\int \cos u \cdot 2du = 2\sin u + C = 2\sin\sqrt{x+1} + C$

—done with the indefinite integral!

So $$I = 2\sin\sqrt{x+1}\Big|_0^8 = 2\sin 3 - 2\sin 1.$$ □

(Same) Example 1, Method II:

same substitution, but change limits in the Definite Integral!

$$I = \int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,dx = \int_{x=0}^{x=8} 2\cos u\,du,$$

but this integral is in terms of u, so need limits of integration for u:

$u = \sqrt{1+x}$

so $x = 0$ gives $u = \sqrt{1+0} = 1$, and $x = 8$ gives $u = \sqrt{1+8} = 3$.

Now $I = \int_{u=1}^{u=3} 2\cos u\,du = 2\sin u\Big|_{u=1}^{u=3} = 2\sin 3 - 2\sin 1.$ $\square$

Example 2: $I = \int_0^{\pi} \left(2 + \sin\left(\frac{x}{2}\right)\right)^2 \cdot \cos\left(\frac{x}{2}\right) \cdot dx.$

S: Use $u = 2 + \sin\left(\frac{x}{2}\right)$

so $du = \left(2 + \sin\left(\frac{x}{2}\right)\right)' dx = \frac{1}{2} \cos\left(\frac{x}{2}\right) \cdot dx.$

Limits: $x = 0 \Rightarrow u = 2 + \sin 0 = 2, \quad x = \pi \Rightarrow u = 2 + \sin \frac{\pi}{2} = 3.$

Now, $I = \int_{u=2}^{u=3} u^2 \cdot 2du = \frac{2}{3} u^3 \Big|_{u=2}^{u=3} = \frac{2}{3}\left(3^3 - 2^3\right) = \frac{38}{3}.$ □

§6.3.1

(i) $\int \tan x \, dx = \ln|\sec x| + C$; (ii) $\int \cot x \, dx = \ln|\sin x| + C$;

(iii)* $\int \sec x \, dx = \ln|\sec x + \tan x| + C$.

Proof (i): $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

so substitute $u = \cos x$ with $du = (\cos x)' \, dx = -\sin x \, dx$.

Now we get:

$\int \frac{1}{u} (-du) = -\ln|u| + C = -\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C.$

□

Proof (ii): Similar, only use $\cot x = \frac{\cos x}{\sin x}$ and $u = \sin x$... (exercise!)

Proof (iii): More complicated, will be given later...

§6.3.2 Integrals of Powers of cos and sin

1. $\boxed{\int \sin^n x \cdot \cos^{2k+1} x \cdot dx} = \int \sin^n x \cdot \big(\underbrace{\cos^2 x}_{1-\sin^2 x}\big)^k \cdot \underbrace{\cos x \cdot dx}_{(\sin x)' \cdot dx}$

$$= \int \sin^n x \cdot \left(1 - \sin^2 x\right)^k \cdot (\sin x)' \cdot dx = \int u^n (1-u^2)^k du, \text{ where } u = \sin x.$$

2. $\boxed{\int \sin^{2k+1} x \cdot \cos^n x \cdot dx}$—similarly, but use $\sin^2 x = 1 - \cos^2 x$ and

$u = \cos x$ with $du = -\sin x\, dx \quad \Rightarrow \quad I = -\int u^n (1-u^2)^k du$...

3. $\boxed{\int \sin^{2k} x \cdot \cos^{2n} x \cdot dx}$ S: use double angle identities:

$$\boxed{\sin^2 x = \tfrac{1}{2}\big(1 - \cos(2x)\big), \quad \cos^2 x = \tfrac{1}{2}\big(1 + \cos(2x)\big)}:$$

Example (third type):

$$\int \sin^4 x \, dx = \int \left(\sin^2 x\right)^2 dx = \int \tfrac{1}{4}\left(1 - \cos(2x)\right)^2 dx$$

$$= \tfrac{1}{4} \int \left(1 - 2\cos(2x) + \cos^2(2x)\right) dx$$

$$= \tfrac{x}{4} - \tfrac{1}{4}\sin(2x) + \tfrac{1}{4} \int \tfrac{1}{2}\left(1 + \cos(4x)\right) dx$$

$$= \tfrac{x}{4} - \tfrac{1}{4}\sin(2x) + \tfrac{1}{8}\left(x + \tfrac{1}{4}\sin(4x)\right) + C. \qquad \square$$

§6.3.3 Inverse Trigonometric Integrals

$$\text{Recall that } \frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}$$

$$\text{and } \frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2-x^2}} \quad (a > 0)$$

$$\text{and also } \frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2+x^2}$$

Hence, $\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C = -\cos^{-1}\left(\frac{x}{a}\right) + C'}$ (with $a > 0$)

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

Examples:

1. $I = \int \frac{t}{\sqrt{1-t^4}}\, dt,$

S: Use $u = t^2$ with $du = (t^2)'\, dt = 2t\, dt$.

$$I = \int \frac{1}{\sqrt{1-u^2}} \cdot \frac{1}{2} du = \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{2} \sin^{-1} u + C = \frac{1}{2} \sin^{-1}(t^2) + C \qquad \square$$

Examples:

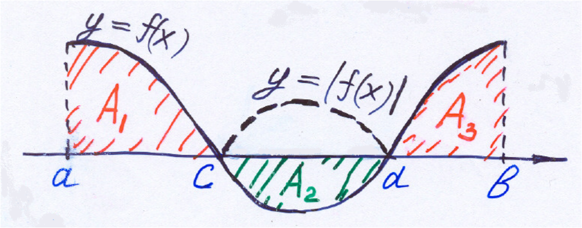
2. $I = \int \frac{dx}{x^2+4x+5}$.

Note: $x^2 + 4x + 5 = (x+2)^2 + 1$.

$$\Rightarrow \; u = x + 2, \quad du = dx, \quad x^2 + 4x + 5 = u^2 + 1,$$

$$I = \int \frac{du}{u^2+1} = \tan^{-1} u + C = \tan^{-1}(x+2) + C. \qquad \square$$

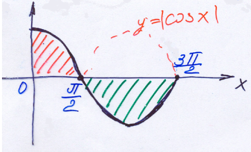
Recall that $\int_a^b f(x)\,dx$ is a signed area: $\int_a^b f(x)\,dx = A_1 - A_2 + A_3$.



The actual area is $A_1 + A_2 + A_3 = \int_a^b |f(x)|\,dx$

$$= \int_a^c f(x)\,dx - \int_c^d f(x)\,dx + \int_d^b f(x)\,dx.$$

Example: Find the area bounded by $y = \cos x$, $y = 0$, $x = 0$, $x = \frac{3\pi}{2}$.



$$A = \int_0^{\frac{3\pi}{2}} |\cos x|\, dx$$

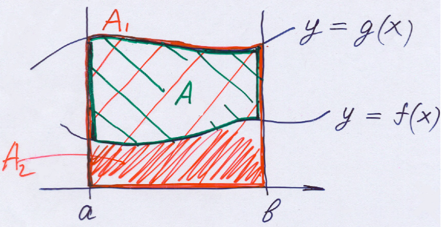
$$|\cos x| = \begin{cases} \cos x, & \text{for } 0 \leqslant x \leqslant \frac{\pi}{2}, \\ -\cos x, & \text{for } \frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}. \end{cases}$$

$$A = \int_0^{\frac{\pi}{2}} \cos x\, dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\, dx = \sin x\Big|_0^{\frac{\pi}{2}} - \sin x\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \left(\sin\tfrac{\pi}{2} - \sin 0\right) - \left(\sin\tfrac{3\pi}{2} - \sin\tfrac{\pi}{2}\right) = (1-0) - (-1-1) = 1 + 2 = 3$$

—both terms 1 and 2 are areas, so both are positive. □

Problem: Find the area bounded by $y = g(x)$, $y = f(x)$, $x = a$, $x = b$.

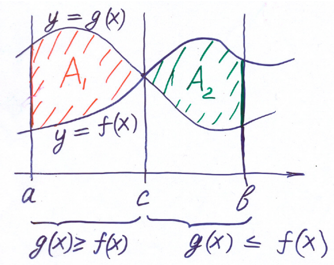


(i) If $g(x) \geqslant f(x)$ on $[a, b]$

$$\Rightarrow \quad A = A_1 - A_2 = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx$$

$$\boxed{A = \int_a^b (g(x) - f(x))\,dx = \int_a^b (\text{upper} - \text{lower})\,dx}$$

Problem: Find the area bounded by $y = g(x)$, $y = f(x)$, $x = a$, $x = b$.



(ii) General case: $\Rightarrow \quad A = A_1 + A_2$

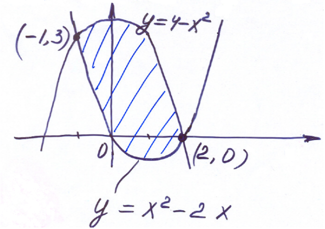
$$= \int_a^c \big(g(x) - f(x)\big)\, dx + \int_c^b \big(f(x) - g(x)\big)\, dx$$

$$= \int_a^c \big(g(x) - f(x)\big)\, dx - \int_c^b \big(g(x) - f(x)\big)\, dx$$

$$\Rightarrow \boxed{A = \int_a^c \big|g(x) - f(x)\big|\, dx}$$

Example 1:

Find the area between the curves $y = x^2 - 2x$ and $y = 4 - x^2$.



Intersections: $x^2 - 2x = 4 - x^2$

$\Rightarrow\ 2x^2 - 2x - 4 = 0$

$\Rightarrow\ x^2 - x - 2 = 0 \quad \Rightarrow x = -1,\ 2.$

Area: $A = \int_{-1}^{2} (\text{upper} - \text{lower})\, dx$

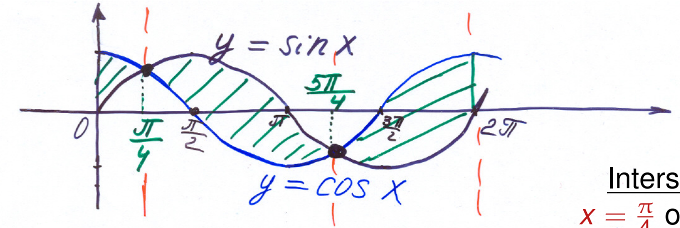
$$= \int_{-1}^{2} \big((4 - x^2) - (x^2 - 2x)\big)\, dx$$

$$= \int_{-1}^{2} (4 - 2x^2 + 2x)\, dx = \left(4x - 2\,\tfrac{x^3}{3} + 2\,\tfrac{x^2}{2}\right)\Big|_{-1}^{2}$$

$$= \left(4 \cdot 2 - \tfrac{2}{3}2^3 + 2^2\right) - \left(-4 - \tfrac{2}{3} + 1\right) = 9.$$

(Note that A is positive!) $\square$

Example 2: Find the total area between the curves $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = 2\pi$.



Intersections:
$x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$.

$$A = \int_0^{2\pi} \left|\cos x - \sin x\right| dx$$

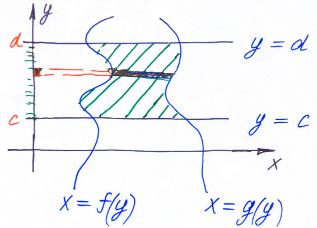
$$= \int_0^{\frac{\pi}{4}} \left(\cos x - \sin x\right) dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(\sin x - \cos x\right) dx + \int_{\frac{5\pi}{4}}^{2\pi} \left(\cos x - \sin x\right) dx$$

$$= \underbrace{\cdots}_{\text{Ex.}} = (\sqrt{2} - 1) + (\sqrt{2} + \sqrt{2}) + (1 + \sqrt{2}) = 4\sqrt{2}.$$

(Note that each of the 3 terms is positive!) □

Sometimes curves are expressed in the form $x = f(y)$:

Find the area between $x = f(y)$ and $x = g(y)$ for $c \leqslant y \leqslant d$.

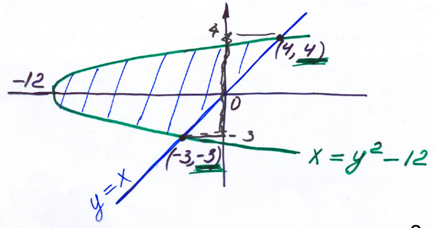


If $g(y) \geqslant f(y)$:

$$A = \int_c^d (\text{right} - \text{left})\, dy = \int_c^d \big(g(y) - f(y)\big)\, dy$$

Generally: $$A = \int_c^d \big|g(y) - f(y)\big|\, dy$$

Example: Find the area between $x = y^2 - 12$ and $y = x$.



Intersection points: $y^2 - 12 = y$

$\Rightarrow\ y^2 - y - 12 = 0 \quad \Rightarrow\ y = 4$ and $y = -3$

—these limits are y-limits.

Area: $A = \int_{-3}^{4} (\text{right} - \text{left})\, dy = \int_{-3}^{4} (y - (y^2 - 12))\, dy$

$= \int_{-3}^{4} (12 + y - y^2)\, dy = \left(12y + \frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_{y=-3}^{4} = \frac{343}{6}.$ □

Recall the Product Rule:

$$\frac{d}{dx}\underbrace{(u\cdot v)} = u\,\frac{dv}{dx} + v\,\frac{du}{dx} \quad\Rightarrow\quad u\cdot v = \int\left(u\,\frac{dv}{dx} + v\,\frac{du}{dx}\right)dx + \underbrace{C}$$

i.e. this is an antiderivative of the right-hand side

skip it since $\int$ implies an arbitrary constant

$$\Rightarrow\quad u\cdot v = \int u\,\underbrace{\frac{dv}{dx}\,dx}_{dv} + \int v\,\underbrace{\frac{du}{dx}\,dx}_{du} \quad\Rightarrow\quad u\cdot v = \int u\,dv + \int v\,du$$

$$\Rightarrow\quad \boxed{\int u\,dv = u\cdot v - \int v\,du}$$

Note:

–we are "trading" one integral $\int u\,dv$ for another $\int v\,du$ (hopefully simpler).

–we use $u \to du$ (differentiation); $dv \to v$ (integration).

§8.2 Examples

1. $\int \underbrace{x}_{=u} \underbrace{e^x\,dx}_{=dv}$ Use: $\left.\begin{array}{l} u = x \\ dv = e^x\,dx \end{array}\right\} \Rightarrow \left\{\begin{array}{l} du = dx \\ v = e^x \end{array}\right.$

$$I = u\,v - \int v\,du = x\,e^x - \underbrace{\int e^x\,dx}_{\text{simpler!}} = x\,e^x - e^x + \underbrace{C}_{\text{don't foget!}} \qquad \square$$

2. $\int \underbrace{\ln x}_{=u} \underbrace{dx}_{=dv}$ Use: $\left.\begin{array}{l} u = \ln x \\ dv = dx \end{array}\right\} \Rightarrow \left\{\begin{array}{l} du = \frac{1}{x}\,dx \\ v = x \end{array}\right.$

$$I = u\,v - \int v\,du = (\ln x)\cdot x - \int x \cdot \frac{1}{x}\,dx = x\,\ln x - x + C. \qquad \square$$

3. $\int \underbrace{x^2}_{=u} \underbrace{\sin x\, dx}_{=dv}$ Use: $\left.\begin{aligned} u &= x^2 \\ dv &= \sin x\, dx \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} du &= 2x\, dx \\ v &= -\cos x \end{aligned}\right.$

$$I = uv - \int v\, du = x^2(-\cos x) - \int(-\cos x)(2x\, dx) = -x^2 \cos x + 2 \underbrace{\int x \cos x\, dx}_{\text{simpler as } x^1 \text{ here}}.$$

$\Rightarrow$ Need another integration by parts:

$\int \underbrace{x}_{=u} \underbrace{\cos x\, dx}_{=dv}$ Use: $\left.\begin{aligned} u &= x \\ dv &= \cos x\, dx \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} du &= dx \\ v &= \sin x \end{aligned}\right.$

$$\int x \cos x\, dx = u\, v - \int v\, du = x \cdot \sin x - \int \sin x\, dx = x \sin x + \cos x + C.$$

So $I = \int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + \underbrace{2C}_{=C'}.$ □

4. $\int x \underbrace{\tan^{-1} x}_{=u}\, dx$ **Use:** $\left.\begin{aligned} u &= \tan^{-1} x \\ dv &= x\,dx \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} du &= \frac{1}{x^2+1}\,dx \\ v &= \frac{x^2}{2} \end{aligned}\right.$

(Note: one can "simplify" $\tan^{-1}$ (also ln, $\sin^{-1}$...) by differentiating it!)

$$I = u\,v - \int v\,du = (\tan^{-1} x)\cdot \frac{x^2}{2} - \int \frac{x^2}{2}\,\frac{1}{x^2+1}\,dx = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \underbrace{\frac{x^2}{x^2+1}}\,dx$$

here $x^2 = (x^2+1) - 1$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int\left(1 - \frac{1}{x^2+1}\right) dx$$

$$= \frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + C. \qquad \square$$

5. $\int \underbrace{\sin^{-1}x}_{=u} \underbrace{dx}_{=dv}$ **Use:** $\left.\begin{aligned} u &= \sin^{-1}x \\ dv &= dx \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} du &= \frac{1}{\sqrt{1-x^2}}\,dx \\ v &= x \end{aligned}\right.$

$$I = u\,v - \int v\,du = (\sin^{-1}x)\cdot x - \underbrace{\int x\,\frac{1}{\sqrt{1-x^2}}\,dx}$$

substitution $w = 1 - x^2$

$dw = -2x\,dx \Rightarrow x\,dx = -\frac{1}{2}dw$

$$= x\,\sin^{-1}x - \int \frac{-\frac{1}{2}\,dw}{\sqrt{w}}$$

$$= x\,\sin^{-1}x + \sqrt{w} + C$$

$$= x\,\sin^{-1}x + \sqrt{1-x^2} + C. \qquad \square$$

§8.3 Original Integral Reappears on the RHS...

1. $\int \underbrace{\sec^3 x\, dx}_{u,\ dv=??}$ Use: $\left.\begin{array}{l} u = \sec x \\ dv = \sec^2 x\, dx \end{array}\right\} \Rightarrow \left\{\begin{array}{l} du = \sec x \tan x\, dx \\ v = \tan x \end{array}\right.$

(Recall that $\sec x = \frac{1}{\cos x}$.)

$$I = u\,v - \int v\,du = \sec x \cdot \tan x - \int \sec x \cdot \underbrace{\tan^2 x}_{(\sec^2 x - 1)} \cdot dx$$

$$= \sec x \cdot \tan x - \int \sec^3 x\, dx + \underbrace{\int \sec x\, dx}_{= \ln|\sec x + \tan x| \text{ (Lecture 6)}}$$

$$= \sec x \cdot \tan x - \underbrace{\int \sec^3 x\, dx}_{=I} + \ln|\sec x + \tan x| + C.$$

So we got: $I = \sec x \cdot \tan x - I + \ln|\sec x + \tan x| + C$

$\Rightarrow \quad 2 \cdot I = \sec x \cdot \tan x + \ln|\sec x + \tan x| + C$

$\Rightarrow \quad I = \frac{1}{2} \sec x \cdot \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C'$

$$\boxed{\int \sec^3 x\, dx = \frac{1}{2} \sec x \cdot \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C}$$

□

2. $\int \underbrace{e^{3x}}_{=u} \underbrace{\cos(2x)\, dx}_{=dv}$ Use: $\left. \begin{aligned} u &= e^{3x} \\ dv &= \cos(2x)\, dx \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} du &= 3\, e^{3x} dx \\ v &= \tfrac{1}{2} \sin(2x) \end{aligned} \right.$

$$I = u\, v - \int v\, du = e^{3x} \cdot \tfrac{1}{2} \sin(2x) - \int \tfrac{1}{2} \sin(2x) \cdot 3\, e^{3x} dx$$

$$\Rightarrow \quad I = \tfrac{1}{2}\, e^{3x} \sin(2x) - \tfrac{3}{2} \int e^{3x} \sin(2x)\, dx \qquad (*)$$

For $\int \underbrace{e^{3x}}_{=u} \underbrace{\sin(2x)\, dx}_{=dv}$ use: $\left. \begin{aligned} u &= e^{3x} \\ dv &= \sin(2x)\, dx \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} du &= 3\, e^{3x} dx \\ v &= -\tfrac{1}{2} \cos(2x) \end{aligned} \right.$

$$\begin{aligned} \int e^{3x} \sin(2x)\, dx &= u\, v - \int v\, du \\ &= e^{3x} \cdot \left(-\tfrac{1}{2} \cos(2x)\right) - \int \left(-\tfrac{1}{2} \cos(2x)\right) \cdot 3\, e^{3x} dx \\ &= -\tfrac{1}{2}\, e^{3x} \cos(2x) + \tfrac{3}{2} \underbrace{\int e^{3x} \cos(2x)\, dx}_{=I} = -\tfrac{1}{2}\, e^{3x} \cos(2x) + \tfrac{3}{2}\, I. \end{aligned}$$

---

2. (cont.)

Substitute in $(*)$: $\Rightarrow \quad I = \frac{1}{2} e^{3x} \sin(2x) - \frac{3}{2} \left(-\frac{1}{2} e^{3x} \cos(2x) + \frac{3}{2} I \right)$

$$\Rightarrow \underbrace{\left(1 + \frac{9}{4}\right)}_{= \frac{13}{4}} I = \frac{1}{2} e^{3x} \sin(2x) + \frac{3}{4} e^{3x} \cos(2x),$$

$$\Rightarrow \text{ Answer: } \boxed{I = \frac{2}{13} e^{3x} \sin(2x) + \frac{3}{13} e^{3x} \cos(2x) + C}. \quad \square$$

....

Remark: combining this result with $(*)$, one gets

$$\int e^{3x} \sin(2x)\, dx = \frac{1}{3} e^{3x} \sin(2x) - \frac{2}{3} I$$

$$= \frac{3}{13} e^{3x} \sin(2x) - \frac{2}{13} e^{3x} \cos(2x) + C'.$$

§8.4 Integration by Parts for Definite Integrals

Include the limits and evaluation symbol: $\int_a^b u \frac{dv}{dx}\, dx = u\, v\Big|_a^b - \int_a^b v \frac{du}{dx}\, dx$

Examples:

1. $\int_0^\pi \underbrace{x}_{=u} \underbrace{\sin x\, dx}_{=dv}$ Use: $\left.\begin{array}{l} u = x \\ dv = \sin x\, dx \end{array}\right\} \Rightarrow \left\{\begin{array}{l} du = dx \\ v = -\cos x \end{array}\right.$

$$I = u\, v\Big|_0^\pi - \int_0^\pi v\, du = x\,(-\cos x)\Big|_0^\pi - \int_0^\pi (-\cos x)\, dx$$

$$= \big(\pi\,(-\cos\pi) - 0\big) + \int_0^\pi \cos x\, dx$$

$$= \pi + \sin x\Big|_0^\pi = \pi + (\sin\pi - \sin 0) = \pi + (0 - 0) = \pi. \qquad \square$$

2. $\int_1^e x^3 \underbrace{(\ln x)^2}_{=u} dx$ Use: $\left.\begin{array}{ll} u &= (\ln x)^2 \\ dv &= x^3 dx \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} du &= 2(\ln x)\frac{1}{x}dx \\ v &= \frac{x^4}{4} \end{array}\right.$

$$I = u\,v\Big|_1^e - \int_1^e v\,du = (\ln x)^2\,\frac{x^4}{4}\Big|_1^e - \int_1^e \frac{x^4}{4}\,2(\ln x)\frac{1}{x}\,dx = \frac{1^2 e^4}{4} - \frac{1}{2}\int_1^e x^3 \underbrace{\ln x}_{=u}\,dx$$

Use: $\left.\begin{array}{ll} u &= \ln x \\ dv &= x^3 dx \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} du &= \frac{1}{x}dx \\ v &= \frac{x^4}{4} \end{array}\right.$

$$\Rightarrow I = \frac{e^4}{4} - \frac{1}{2}\left(\ln x\,\frac{x^4}{4}\Big|_1^e - \int_1^e \frac{x^4}{4}\,\frac{1}{x}\,dx\right)$$

$$= \frac{e^4}{4} - \frac{1}{2}\left(\frac{1\,e^4}{4} - \frac{1}{4}\int_1^e x^3 dx\right) = \frac{e^4}{4} - \frac{e^4}{8} + \frac{1}{8}\int_1^e x^3 dx = \frac{e^4}{8} + \frac{1}{8}\,\frac{x^4}{4}\Big|_1^e$$

$$= \frac{e^4}{8} + \frac{1}{8}\left(\frac{e^4}{4} - \frac{1}{4}\right) = \frac{5}{32}e^4 - \frac{1}{32}. \qquad \square$$