1. Example: We want to evaluate an integral with a parameter $n \in \{0, 1, 2 \cdots\}$, e.g., $\boxed{I_n = \int x^n e^x \, dx}$.

Integrate by parts using: $\left.\begin{array}{rl} u & = x^n \\ dv & = e^x \, dx \end{array}\right\} \Rightarrow \left\{\begin{array}{rl} du & = n\, x^{n-1} dx \\ v & = e^x \end{array}\right.$

$$I_n = u\,v - \int v\,du = x^n e^x - n \underbrace{\int x^{n-1} e^x \, dx}_{= I_{n-1}} \Rightarrow \boxed{I_n = x^n e^x - n\, I_{n-1}} \; (n \geqslant 1).$$

For example, $\int x^4 e^x \, dx = I_4 = x^4 e^x - 4\, I_3$

$$= x^4 e^x - 4\left(x^3 e^x - 3\, I_2\right) = x^4 e^x - 4x^3 e^x + 12\left(x^2 e^x - 2\, I_1\right)$$

$$= x^4 e^x - 4x^3 e^x + 12\, x^2 e^x - 24\left(x^1 e^x - 1\, I_0\right).$$

Here $I_0 = \int x^0 e^x \, dx = \int e^x \, dx = e^x + C$.

$$\Rightarrow \int x^4 e^x \, dx = \left(x^4 - 4x^3 + 12x^2 - 24x + 24\right) e^x + C'. \quad \square$$

Example: Find an iterative reduction formula for $\boxed{I_n = \int_0^{\pi/2} \cos^n x\, dx}$.

$$I_0 = \int_0^{\pi/2} 1 \cdot dx = \frac{\pi}{2}, \qquad I_1 = \int_0^{\pi/2} \cos x \cdot dx = \sin x\Big|_0^{\pi/2} = 1.$$

Now assume: $n \geqslant 2$ and focus on $\quad I_n = \displaystyle\int_0^{\pi/2} \underbrace{\cos^{n-1} x}_{=u}\, \underbrace{\cos x\, dx}_{=dv}$

Use: $\left.\begin{aligned} u &= \cos^{n-1} x \\ dv &= \cos x\, dx \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} du &= (n-1)\cos^{n-2} x \cdot (-\sin x) dx \\ v &= \sin x \end{aligned}\right.$

$$\begin{aligned} I_n &= u\, v\Big|_0^{\pi/2} - \int_0^{\pi/2} v\, du \\ &= \underbrace{\cos^{n-1} x \cdot \sin x\Big|_0^{\pi/2}}_{\text{Note: } \sin 0 = 0,\ \cos \frac{\pi}{2} = 0} - \int_0^{\pi/2} \sin x \cdot (n-1)\cos^{n-2} x \cdot (-\sin x)\, dx \\ &= (0-0) + (n-1)\int_0^{\pi/2} \underbrace{\sin^2 x}_{(1-\cos^2 x)} \cdot \cos^{n-2} x \cdot dx \\ &= (n-1)\left(\int_0^{\pi/2} \cos^{n-2} x\, dx - \int_0^{\pi/2} \cos^n x\, dx\right) = (n-1)\left(I_{n-2} - I_n\right). \end{aligned}$$

So we got: $\quad I_n = (n-1)\left(I_{n-2} - I_n\right)$ for $n \geqslant 2$,

$$\Rightarrow \; n\, I_n = (n-1)\, I_{n-2} \Rightarrow \boxed{\int_0^{\pi/2} \cos^n x\, dx = I_n = \left(\frac{n-1}{n}\right) I_{n-2}} \text{ for } n \geqslant 2. \quad \square$$

E.g.,

$$\int_0^{\pi/2} \cos^5 x\, dx = I_5 = \frac{4}{5}\, I_3 = \frac{4}{5}\cdot\frac{2}{3}\, I_1 = \frac{4}{5}\cdot\frac{2}{3}\cdot 1 = \frac{8}{15}.$$

$$\int_0^{\pi/2} \cos^6 x\, dx = I_6 = \frac{5}{6}\, I_4 = \frac{5}{6}\cdot\frac{3}{4}\, I_2 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\, I_0 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}.$$

In general,

$$I_{2k} = \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdot\frac{2k-5}{2k-4}\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{(2k)!\,\pi}{(k!)^2\, 2^{2k+1}};$$

$$I_{2k+1} = \frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}\cdot\frac{2k-4}{2k-3}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1 = \frac{(k!)^2\, 4^k}{(2k+1)!}.$$

§9.2 Inverse Trigonometric Substitutions

What is an inverse substitution?—So far, a substitution was defined as a function of x: $u = u(x)$ with $du = u'(x)\,dx$. Now, consider an inverse substitution $\boxed{x = g(u)}$ so

$$\int_a^b f(x)\,dx = \int_{x=a}^{x=b} f\big(g(u)\big)\,\underbrace{g'(u)\,du}_{dx}.$$

Integral involves	Inverse Trigonometric Substitution	Note
$\sqrt{a^2 - x^2}$	(1) Sin substitution $x = a\sin\theta$	$\theta = \sin^{-1}\frac{x}{a}$
$\sqrt{a^2 + x^2}$	(2) Tangent substitution $x = a\tan\theta$	$\theta = \tan^{-1}\frac{x}{a}$
$\sqrt{x^2 - a^2}$	(3)* Secant substitution $x = a\sec\theta$	$\theta = \sec^{-1}\frac{x}{a}$

(Here $a > 0$!)

Remark:
these substitutions are worth trying, but there is no guarantee!

Motivation for the 3 substitutions:

1. $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = a \cos\theta,$ where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}],$

   $dx = a\cos\theta \cdot d\theta.$

2. $\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2 \tan^2\theta} = a \sec\theta,$ where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}),$

   $dx = a\sec^2\theta \cdot d\theta.$

3. Be more careful!

$$\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2\theta - a^2} = \begin{cases} a\tan\theta & \text{for } x \geqslant a, \quad \theta \in [0, \frac{\pi}{2}), \\ -a\tan\theta & \text{for } x \leqslant -a, \quad \theta \in (\frac{\pi}{2}, \pi], \end{cases}$$

$$dx = d(a\sec\theta) = a\sec\theta\tan\theta \cdot d\theta.$$

Examples:

1. $I = \int \frac{dx}{(5-x^2)^{3/2}}$: $\quad x = \sqrt{5}\sin\theta, \quad dx = \sqrt{5}\cos\theta\, d\theta,$

$$(5 - x^2)^{3/2} = (5 - 5\sin^2\theta)^{3/2} = (\sqrt{5}\cos\theta)^3,$$

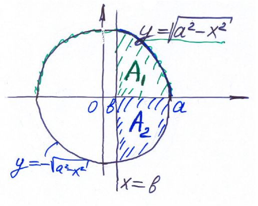
$$\Rightarrow \quad I = \int \frac{\sqrt{5}\cos\theta\, d\theta}{(\sqrt{5}\cos\theta)^3} = \int \frac{d\theta}{5\cos^2\theta} = \frac{1}{5}\tan\theta + C \underbrace{=}_{\text{Don't forget to return to the original variable } x!} \frac{1}{5}\frac{\sin\theta}{\cos\theta} + C$$

Don't forget to return to the original variable x!

$$= \frac{1}{5}\frac{\frac{x}{\sqrt{5}}}{\sqrt{1-\left(\frac{x}{\sqrt{5}}\right)^2}} + C = \frac{1}{5}\frac{x}{\sqrt{5-x^2}} + C.$$

□

2. Find the area enclosed between $x^2 + y^2 = a^2$ and $x = b$ to the right of $x = b$ (with $0 < b < a$).



$$A = A_1 + A_2 = 2A_1 = 2\int_b^a \sqrt{a^2 - x^2}\, dx,$$

$x = a \sin\theta$,
$dx = a\cos\theta\, d\theta, \quad \sqrt{a^2 - x^2} = a\cos\theta,$

$$A = 2\int_{x=b}^{x=a} a^2 \underbrace{\cos^2\theta}_{\frac{1}{2}(1+\cos(2\theta))}\, d\theta = a^2\Big(\underbrace{\theta}_{\sin^{-1}\frac{x}{a}} + \underbrace{\tfrac{1}{2}\sin(2\theta)}_{\sin\theta\cdot\cos\theta = \frac{x}{a}\sqrt{1-(\frac{x}{a})^2}}\Big)\Big|_{x=b}^{x=a}$$

$$= \left(a^2 \sin^{-1}\frac{x}{a} + x\sqrt{a^2 - x^2}\right)\Big|_{x=b}^{x=a}$$

$$= a^2 \sin^{-1} 1 - a^2 \sin^{-1}\frac{b}{a} - b\sqrt{a^2 - b^2}$$

$$= a^2\frac{\pi}{2} - a^2 \sin^{-1}\frac{b}{a} - b\sqrt{a^2 - b^2}. \qquad \square$$

Note: for $b = a$ we get $A = 0$, while $b = 0$ yields $A = a^2\frac{\pi}{2}$ (as expected!)

3. $I = \int \frac{dx}{\sqrt{4+x^2}}$: $\quad x = 2\tan\theta \quad \Rightarrow dx = 2\sec^2\theta\, d\theta,$

$$\sqrt{4+x^2} = \sqrt{4+4\tan^2\theta} = \sqrt{4\sec^2\theta} = 2\sec\theta.$$

$$\Rightarrow \quad I = \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int \sec\theta\, d\theta = \ln\Big|\underbrace{\sec\theta}_{=\sqrt{1+\tan^2\theta}=\sqrt{1+\frac{x^2}{4}}} + \underbrace{\tan\theta}_{=\frac{x}{2}}\Big| + C$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C = \ln\big(\underbrace{\sqrt{4+x^2}+x}_{>0}\big) + C',$$

(where we used $\ln|\frac{a}{2}| = \ln|a| - \ln 2$ so $C' = C - \ln 2$). □

4. $I = \int \frac{dx}{x\sqrt{x^2-a^2}}$, where $x \geqslant a > 0$:

$x = a\sec\theta \quad \Rightarrow dx = a\sec\theta\,\tan\theta\,d\theta,$

$$\sqrt{x^2-a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \underbrace{+a\tan\theta}_{\Leftarrow x \geqslant a > 0}.$$

$$\Rightarrow \quad I = \int \frac{a\sec\theta\,\tan\theta\,d\theta}{(a\sec\theta)(a\tan\theta)} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$$

for $x \geqslant a > 0$. □

EX: solve this for $x \leqslant -a < 0$.

§9.3 Completing the Square

If an integral involves a quadratic expression:

$$\boxed{ax^2+bx+c} = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) = a\underbrace{\left(x^2+2\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2\right)}_{=\left(x+\frac{b}{2a}\right)^2\text{—completing the square!}} - \frac{b^2}{4}+c$$

$$= \left(x+\frac{b}{2a}\right)^2 - \underbrace{\frac{b^2}{4}+c}_{\text{constant!}} \Rightarrow \text{ substituion } \boxed{u = x+\frac{b}{2a}}\ldots$$

Examples

1. $I = \int \frac{dx}{\sqrt{2x - x^2}}$:

$$2x - x^2 = -(x^2 - 2x) = -(x^2 - 2x + 1) + 1 = 1 - (x - 1)^2,$$

$$\Rightarrow u = x - 1,\ du = dx,\ 2x - x^2 = 1 - u^2,$$

Now $I = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(x - 1) + C.$ □

2. $I = \int \frac{dx}{x^2+2x+10}$

$$\underbrace{x^2+2x}+10 = (x^2+2x+1)-1+10 = (x+1)^2+9$$

$$\Rightarrow u = x+1,\ du = dx,\ x^2+2x+10 = u^2+9,$$

Now $I = \int \frac{du}{u^2+9} = \frac{1}{3}\tan^{-1}\frac{u}{3} + C = \frac{1}{3}\tan^{-1}\frac{x+1}{3} + C.$ □

Integral involves:	Substitution worth trying:
$\sqrt{ax+b}$	$ax+b=u^2 \;\Rightarrow\; dx=\frac{2}{a}\,u\,du$
$\cos\theta$ and/or $\sin\theta$	$x=\tan\frac{\theta}{2} \;\Rightarrow\; \sin\theta=\frac{2x}{1+x^2},\quad \cos\theta=\frac{1-x^2}{1+x^2}$

$\int \frac{P(x)}{Q(x)} dx$, where $P(x)$ and $Q(x)$ are polynomials.

Here $\frac{P(x)}{Q(x)}$ is called a <u>rational function</u>.

§10.1 Linear Denominator

If $Q(x) = ax + b \quad \Rightarrow \quad$ Substitution $\boxed{u = ax + b}$

Example: $\displaystyle\int \frac{x^2+3}{2x-1}dx \quad \Rightarrow \quad u = 2x - 1$

$$\Rightarrow \quad x = \tfrac{u+1}{2}, \quad dx = \tfrac{1}{2}du, \quad x^2 + 3 = \left(\tfrac{u+1}{2}\right)^2 + 3 = \tfrac{1}{4}(u^2 + 2u + 13).$$

$$\Rightarrow \; I = \int \tfrac{u^2+2u+13}{4u} \cdot \tfrac{1}{2}du = \tfrac{1}{8}\int \left(u+2+\tfrac{13}{u}\right)du = \tfrac{1}{8}\left(\tfrac{1}{2}u^2+2u+13\ln|u|\right)+C$$

$$= \tfrac{1}{8}\Big(\underbrace{\tfrac{1}{2}(2x-1)^2 + 2(2x-1)}_{=\frac{1}{2}(4x^2-4x+1)+4x-2=2x^2+2x-\frac{3}{2}} + 13\ln|2x-1|\Big) + C$$

$$= \frac{x^2}{4} + \frac{x}{4} - \frac{3}{16} + \frac{13}{8}\ln|2x-1| + C = \frac{x^2}{4} + \frac{x}{4} + \frac{13}{8}\ln|2x-1| + C'. \qquad \square$$

§10.2 Quadratic Denominators

Consider $Q(x) = ax^2 + bx + c$

Step 0: Let $P(x)$ be an nth degree polynomial.

If $n \geqslant 2 \Rightarrow$ use long division to rewrite:

$$\frac{P(x)}{Q(x)} = \underbrace{R_{n-1}(x)}_{\text{easy to integrate!}} + \underbrace{\frac{Ax+B}{ax^2+bx+c}}_{\text{so focus on this!}}$$

Step 1: Complete the square!

$$Q(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Substitute: $u = x + \frac{b}{2a}$ to get (for some constant α, β and γ):

$$\int \frac{\alpha u + \beta}{u^2+\gamma} du = \alpha \int \frac{u}{u^2+\gamma} + \beta \int \frac{du}{u^2+\gamma}$$

Step 2: Use elementary integrals (from the log tables) such as

$$\int \frac{x}{x^2+\gamma} dx = \frac{1}{2} \ln |x^2 + \gamma| + C \text{ (with } \gamma \neq 0);$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \quad \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C \text{ (with } a > 0);$$

Also $\int \frac{dx}{x} = \ln |x| + C$; $\int \frac{dx}{x^2} = -\frac{1}{x} + C$.

Examples:

1. $\int \frac{x+4}{x^2-5x+6}dx$: **Step 1:** $\underbrace{x^2-5x}_{=x^2-2\frac{5}{2}x\pm\left(\frac{5}{2}\right)^2} +6 = \left(x-\frac{5}{2}\right)^2-\frac{1}{4}$

$$\Rightarrow\ u = x-\frac{5}{2} \quad \Rightarrow \quad du = dx; \qquad x+4 = \left(u+\frac{5}{2}\right)+4$$

$$\Rightarrow \quad I = \int \frac{u+\frac{13}{2}}{u^2-\frac{1}{4}}du = \int \frac{u}{u^2-\frac{1}{4}}du + \frac{13}{2}\int \frac{du}{u^2-\left(\frac{1}{2}\right)^2}.$$

Step 2: $I = \frac{1}{2}\ \ln\Big|\underbrace{u^2-\frac{1}{4}}_{x^2-5x+6=(x-3)(x-2)}\Big| + \frac{13}{2}\,\frac{1}{2\cdot\frac{1}{2}}\ \ln\left|\frac{u-\frac{1}{2}}{u+\frac{1}{2}}\right| + C$

$$= \frac{1}{2}\underbrace{\ln\big|(x-3)(x-2)\big|}_{=\ln|x-3|+\ln|x-2|} + \frac{13}{2}\underbrace{\ln\left|\frac{x-3}{x-2}\right|}_{\ln|x-3|-\ln|x-2|} + C = 7\ln|x-3| - 6\ln|x-2| + C.$$

2. $\int \frac{x+1}{x^2-4x+4}dx$:

Step 1: $x^2-4x+4=(x-2)^2$

$$\Rightarrow \quad u=x-2 \quad \Rightarrow \quad du=dx, \;\; x=u+2$$

$$\Rightarrow \quad I=\int \frac{(u+2)+1}{u^2}du=\int \frac{du}{u}+3\int \frac{du}{u^2}.$$

Step 2: $I=\ln|u|-\frac{3}{u}+C=\ln|x-2|-\frac{3}{x-2}+C.$ □

3. $\int \frac{x^3+3x^2}{x^2+1} dx$:

Step 0:

$$\begin{array}{r|l} & x+3 \\ \hline x^2+1 & x^3+3x^2 \\ & x^3 \qquad +x \\ \hline & \qquad 3x^2-x \\ & \qquad 3x^2 \qquad +3 \\ \hline & \qquad\qquad -x-3 \end{array}$$

So $\frac{x^3+3x^2}{x^2+1} = x+3+\frac{-x-3}{x^2+1}$

Step 1: no need to use any substitutions,

$$I = \int (x+3)\,dx - \int \frac{x}{x^2+1} dx - 3\int \frac{dx}{x^2+1}.$$

Step 2: $I = \left(\frac{x^2}{2}+3x\right) - \frac{1}{2}\ln(x^2+1) - 3\tan^{-1}x + C.$ □

4. $\int \frac{x^3+1}{x^2+7x+12}dx$:

Step 0:

$$\begin{array}{r|l} x^2+7x+12 & \overline{x-7} \\ & x^3 \qquad\qquad +1 \\ & x^3+7x^2+12x \\ \hline & -7x^2-12x+1 \\ & -7x^2-49x-84 \\ \hline & 37x+85 \end{array}$$

So $I=\int\left(x-7+\frac{37x+85}{x^2+7x+12}\right)dx=\frac{x^2}{2}-7x+\int\frac{37x+85}{x^2+7x+12}dx$ (*)

Step 1: $x^2+7x+12=\left(x+\frac{7}{2}\right)^2-\frac{1}{4}$

$\Rightarrow u=x+\frac{7}{2},\quad du=dx,\quad 37x+85=37(u-\frac{7}{2})+85=37u-\frac{89}{2}.$

$$\int\frac{37x+85}{x^2+7x+12}dx=\int\frac{37u-\frac{89}{2}}{u^2-\frac{1}{4}}du=37\int\frac{u}{u^2-\frac{1}{4}}du-\frac{89}{2}\int\frac{du}{u^2-\left(\frac{1}{2}\right)^2}$$

Step 2: $=\frac{37}{2}\ln|\underbrace{u^2-\frac{1}{4}}_{x^2+7x+12=(x+3)(x+4)}|-\frac{89}{2}\ln\Big|\underbrace{\frac{u-\frac{1}{2}}{u+\frac{1}{2}}}_{\frac{x+3}{x+4}}\Big|+C$

$$=\frac{37}{2}\left(\ln|x+3|+\ln|x+4|\right)-\frac{89}{2}\left(\ln|x+3|-\ln|x+4|\right)+C;$$

Now simplify and substitute in (*) to get I (Ex). □

§10.3 General Method: The Method of Partial Fractions

Evaluate $\int \frac{P_m(x)}{Q_n(x)} dx$

Step 0: If $m \geqslant n$ $\Rightarrow$ use long division to rewrite:

$$\frac{P_m(x)}{Q_n(x)} = \underbrace{R(x)}_{\text{easy to integrate!}} + \underbrace{\frac{T_{n-1}(x)}{Q_n(x)}}_{\text{so focus on this!}}$$

Now the problem is reduced to the case $\boxed{m < n}$!

Step 1: Find a partial fraction representation of $\frac{P_m(x)}{Q_n(x)}$ as a sum of terms of type: $\frac{A}{(x-a)^k}$ and $\frac{Ax+B}{(ax^2+bx+c)^k}$ (where $k = 1, 2, \cdots$)

Step 2: Integrate each partial fraction:

$\int \frac{dx}{x-a} = \ln|x-a| + C$; $\int \frac{dx}{(x-a)^k} = \frac{1}{(k-1)(x-a)^{k-1}} + C$ $(k > 1)$;

Also use integrals from §10.2, Step 2; may need to use other elementary integrals...

§10.4 Partial Fractions, Case 1: Q_n has n distinct real roots $a_1, a_2, \cdots, a_n$

Hence, $\boxed{Q_n(x) = (x-a_1)(x-a_2)\cdots(x-a_n)}$

Then Step 1 is

$$\boxed{\frac{P_m(x)}{Q_n(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}}$$ (with $m < n$)

Here the constants $A_1, A_2 \cdots A_n$ are to be computed as follows:

- multiply by $(x-a_1)(x-a_2)\cdots(x-a_n)$;
- successively set $x = a_1,\ x = a_2,\ \cdots,\ x = a_n$.

Example: $I = \int \frac{x^3+2}{x^3-x}dx$

Step 0: Use division $\frac{x^3+2}{x^3-x} = \frac{(x^3-x)+(x+2)}{x^3-x} = 1 + \frac{x+2}{x^3-x}$

$\Rightarrow \; I = x + \int \frac{x+2}{x^3-x}dx.$

Step 1: $x^3 - x = x(x^2-1) = x(x-1)(x+1) \;\Rightarrow$ 3 real roots: $0,\ 1,\ -1$.

Do a partial fraction decomposition: $\boxed{\frac{x+2}{x^3-x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}}$.

- Multiply by $x(x-1)(x+1) = x^3 - x$:
  $x + 2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$;
- Set $x = 0 \;\Rightarrow\; 2 = A(-1)(1) + 0 + 0 \;\Rightarrow\; A = -2$;
- Set $x = 1 \;\Rightarrow\; 3 = 0 + B(1)(2) + 0 \;\Rightarrow\; B = \frac{3}{2}$;
- Set $x = -1 \;\Rightarrow\; 1 = 0 + 0 + C(-1)(-2) \;\Rightarrow\; C = \frac{1}{2}$.

Finally, $\frac{x+2}{x^3-x} = \frac{-2}{x} + \frac{3}{2}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1}$.

Step 2: $I = x - 2\int \frac{dx}{x} + \frac{3}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1}$

$= x - 2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C.$ □

Lecture 11 §11.1 Partial Fractions, Case 2: Q_n has multiple real roots

Recall: if a_1 is a root of $Q_n(x)$ of multiplicity k, then $\boxed{(x-a_1)^k}$ is a factor of $Q_n(x) \Rightarrow$ include k terms in the Partial Fraction Decomposition

$$\Rightarrow \boxed{\frac{A_1}{x-a_1} + \frac{A_2}{(x-a_1)^2} + \cdots + \frac{A_k}{(x-a_1)^k}}.$$

Examples:

1. $I = \int \frac{dx}{x(x-1)^2}$:

   Step 1: $Q(x) = x(x-1)^2$ has a simple root 0 and a double root 1.

   Hence, $\boxed{\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}}$.

   - Multiply by $x(x-1)^2$: $1 = A(x-1)^2 + Bx(x-1) + Cx$;
   - Set $x = 0$: $1 = A(-1)^2 + 0 + 0 \Rightarrow A = 1$;
   - Set $x = 1$: $1 = 0 + 0 + C \Rightarrow C = 1$;
   - What about $B = ??$: e.g., set $x = 2$: $1 = A + 2B + 2C \Rightarrow B = -1$;

   So $I = \int \left(\frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\right) dx$.

   Step 2: $I = \ln|x| - \ln|x-1| - \frac{1}{x-1} + C$. □

2. $I = \int \frac{dx}{x^4 - 3x^3}$:

Step 1: $Q(x) = x^4 - 3x^3 = x^3(x-3) \Rightarrow$ roots: 0, 0, 0, 3.

$$\Rightarrow \boxed{\frac{1}{x^4-3x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-3}}$$

- Multiply by $x^3(x-3)$:
$$\Rightarrow \quad 1 = A\,x^2(x-3) + B\,x(x-3) + C\,(x-3) + D\,x^3.$$
- Set $x = 0$: $1 = 0 + 0 + C\,(-3) + 0 \quad \Rightarrow \quad C = -\frac{1}{3}$;
- Set $x = 3$: $1 = 0 + 0 + 0 + D \cdot 3^3 \quad \Rightarrow \quad D = \frac{1}{27}$;
- To get A and B:

$$1 = A\,(x^3 - 3x^2) + B\,(x^2 - 3x) + C\,(x-3) + D\,x^3$$
$$= x^3\underbrace{(A + D)}_{=0} + x^2\underbrace{(-3A + B)}_{=0} + x\underbrace{(-3B + C)}_{=0}\underbrace{-3C}_{=1}$$

$$\Rightarrow \quad A = -D = -\frac{1}{27}; \quad B = 3A = -\frac{1}{9}.$$

So $\frac{1}{x^4-3x^3} = -\frac{1}{27\,x} - \frac{1}{9\,x^2} - \frac{1}{3\,x^3} + \frac{1}{27} \cdot \frac{1}{x-3}$.

Step 2: $I = -\frac{1}{27}\ln|x| + \frac{1}{9x} + \frac{1}{6x^2} + \frac{1}{27}\ln|x-3| + C.$ □

§11.2 Partial Fractions, Case 3: Q_n has complex roots

Then $Q_n(x)$ has a real factor $(x^2 + ax + b)^k$ (where the roots of $x^2 + ax + b$ are complex).

We restrict ourselves to the case $\boxed{k = 1}$.

$\Rightarrow$ In this case, include in the partial fraction decomposition a term:

$$\boxed{\frac{Ax + B}{x^2 + ax + b}}$$

(where the constants A and B are to be evaluated).

Examples:

1. $I = \int \frac{2+3x+x^2}{x(x^2+1)} dx$: **Step 1:** $Q(x) = x(x^2+1)$ has roots $0, i, -i$,

$$\Rightarrow \boxed{\frac{2+3x+x^2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}} \text{ — Multiply by } x(x^2+1):$$

$$2+3x+x^2 = A(x^2+1) + (Bx+C)x = x^2\underbrace{(A+B)}_{=\ 1} + x\underbrace{C}_{=3} + \underbrace{A}_{=2}$$

$$\Rightarrow B = 1 - A = -1$$

Step 2: $I = \int \left(\frac{2}{x} + \frac{-x+3}{x^2+1}\right) dx$

$$= \int \left(\frac{2}{x} - \underbrace{\frac{x}{x^2+1} + \frac{3}{x^2+1}}_{\text{Recall Step 2 in §10.2}}\right) dx = 2\ln|x| - \frac{1}{2}\ln(x^2+1) + 3\tan^{-1}x + C. \ \square$$

2. $I = \int \frac{dx}{x^3+1}$: Step 1: $Q(x) = x^3 + 1 = (x+1)\underbrace{(x^2 - x + 1)}_{\text{complex roots}}$;

$$\Rightarrow \boxed{\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}} \Rightarrow 1 = A(x^2 - x + 1) + (Bx + C)(x+1).$$

Set $x = -1$: $1 = A \cdot 3 + 0 \quad \Rightarrow \quad A = \frac{1}{3}$;

To find B and C: $1 = x^2 \underbrace{(A+B)}_{=0 \Rightarrow B = -\frac{1}{3}} + x(-A + B + C) + \underbrace{(A+C)}_{=1 \Rightarrow C = \frac{2}{3}}$

So $I = \int \left(\frac{1}{3} \frac{1}{x+1} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1} \right) dx.$

Step 2: $I = \frac{1}{3} \ln|x+1| - \frac{1}{3} \underbrace{\int \frac{x-2}{x^2-x+1} dx}_{\text{see §10.2...}}$

Complete the square: $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$

$$\Rightarrow u = x - \frac{1}{2} \quad \Rightarrow \int \frac{x-2}{x^2-x+1} dx = \int \frac{u - \frac{3}{2}}{u^2 + \frac{3}{4}} du = \int \frac{u}{u^2+\frac{3}{4}} du - \frac{3}{2} \int \frac{du}{u^2 + \frac{3}{4}}$$

$$= \frac{1}{2} \ln|u^2 + \tfrac{3}{4}| - \frac{3}{2} \frac{1}{\sqrt{3}/2} \tan^{-1}\left(\frac{u}{\sqrt{3}/2}\right) + C$$

$$= \frac{1}{2} \ln(x^2 - x + 1) - \sqrt{3} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C.$$

$$\Rightarrow I = \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln(x^2 - x + 1) - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C'. \qquad \square$$

Definition

If f is continuous on $[a, \infty)$, we define the improper integral of f over $[a, \infty)$ as

$$\boxed{\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx}.$$

If f is continuous on $(-\infty, a]$, we define the improper integral of f over $(-\infty, a]$ as

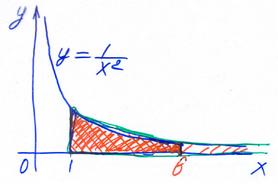
$$\boxed{\int_{-\infty}^{a} f(x)\,dx = \lim_{b\to-\infty} \int_b^a f(x)\,dx}.$$

In either case, if the limit exists as a finite number, the improper integral converges; if the limit does NOT exist, the improper integral diverges.

NOTE: one can recognize Type I by infinite limit(s) of integration.

Examples:

1. (To motivate the above definition) Find the area under the curve $y = \frac{1}{x^2}$ above $y = 0$ to the right from $x = 1$:



Consider $\int_1^b \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^b = \boxed{-\frac{1}{b} + 1}$.

To get the whole area, let $b \to \infty$: $\lim_{b\to\infty} \boxed{-\frac{1}{b} + 1} = 1$.

Thus $A = 1$, and the improper integral $\int_1^\infty \frac{1}{x^2}dx$ converges and $= 1$.□

3. Let $a > 0$. For which values of p, does $\int_a^\infty \frac{1}{x^p} dx$ converge?

S: Consider $\int_a^b \frac{1}{x^p} dx = \int_a^b x^{-p} d = \underbrace{\frac{x^{-p+1}}{-p+1}\Big|_a^b}_{(\text{here } p \neq 1)} = \frac{b^{-p+1} - a^{-p+1}}{-p+1}$.

Note: $\lim_{b\to\infty} b^{-p+1} = \begin{cases} 0, & p > 1 \text{ (i.e. } (-p+1) < 0), \\ \infty, & p < 1 \text{ (i.e. } (-p+1) > 0) \end{cases}$

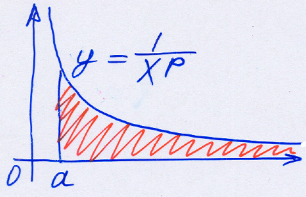
Now $\lim_{b\to\infty} \int_a^b \frac{1}{x^p} dx = \begin{cases} \frac{a^{-p+1}}{p-1}, & p > 1, \\ \text{diverges}, & p < 1. \end{cases}$

If $p = 1$: $\int_a^b \frac{1}{x} dx = \ln x \Big|_a^b = \ln b - \ln a$;

as $\lim_{b\to\infty} \ln b = \infty$, so the integral diverges.

Memorize:

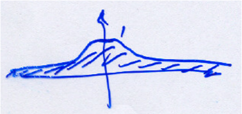
If $p > 1$, then $\int_a^{\infty} \frac{1}{x^p} dx$ converges and equals $\frac{a^{-p+1}}{p-1}$.
If $p \leqslant 1$, then it diverges.



i.e. for $p \leqslant 1$, the area is infinite,

while for $p > 1$ the area $A = \frac{a^{-p+1}}{p-1}$.

3. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \underbrace{\tan^{-1} x\Big|_{-\infty}^{\infty}}_{\text{i.e. } \lim_{a,-b\to\infty} \tan^{-1} x\big|_b^a} = \tan^{-1}(\infty) - \tan^{-1}(-\infty)$



$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$

4. $\underbrace{\int_0^{\infty} e^{-st}\,dt}_{(\text{here } s>0)} = \frac{e^{-st}}{-s}\Big|_{t=0}^{t=\infty} = \frac{e^{-\infty}}{-s} - \frac{e^0}{-s} = 0 + \frac{1}{s} = \frac{1}{s}.$ □

5. $\int_{-\infty}^{2} x\,e^{-x^2}\,dx \underbrace{=}_{u=x^2} \dots = -\frac{1}{2}\,e^{-x^2}\Big|_{x=-\infty}^{x=2} = -\frac{1}{2}\,e^{-4} - \left(-\frac{1}{2}\,e^{-\infty}\right)$

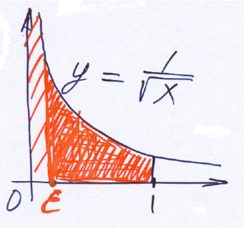
$= -\frac{1}{2}\,e^{-4} - 0 = -\frac{e^{-4}}{2}.$ □

§12.2 Improper Integrals of Type II

such as $\boxed{\int_0^1 \frac{dx}{\sqrt{x}}}$, $\int_0^{\pi} \tan x\,dx$, $\int_{-1}^1 \frac{dx}{e^x - 1}$, $\int_0^{\pi/2} \frac{dx}{\sin x - \cos x}$

—look like proper integrals (as limits of integration are finite!)

But the integrand does NOT exist at $\boxed{x = 0}$, $x = \frac{\pi}{2}$, $x = 0$, $x = \frac{\pi}{4}$, respectively!



E.g. $\frac{1}{\sqrt{x}}$ doesn't exist at $x = 0$.

$\Rightarrow$ The area might be infinite...

Instead consider $\int_\varepsilon^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_\varepsilon^1 = 2 - 2\sqrt{\varepsilon}$ (for very small $\varepsilon > 0$).

Now, $\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{dx}{\sqrt{x}} = 2 \quad \Rightarrow \boxed{\text{area} = 2}$. □

We say: the improper integral of Type II $\int_0^1 \frac{dx}{\sqrt{x}}$ converges and equals 2.

Definition

If $f(x)$ is continuous on $(a, b]$, but $f(a)$ does not exist, we define the improper integral $\boxed{\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx}$.

Similarly, if $f(x)$ is continuous on $[a, b)$, but $f(b)$ does not exist, we define the improper integral $\boxed{\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx}$

Examples:

1. $\int_0^1 \frac{dx}{x} = \lim_{c\to 0^+} \int_c^1 \frac{dx}{x} = \lim_{c\to 0^+} \ln|x|\Big|_c^1 = \lim_{c\to 0^+} (\underbrace{\ln 1}_{=0} - \ln c) = +\infty.$

   So this improper integral diverges. □

2. $\int_0^2 \frac{dx}{\sqrt{2x-x^2}}$: here the integrand blows up at $x = 0, 2$!

   S: Complete the square: $2x - x^2 = -(x-1)^2 + 1$

   $\Rightarrow$ Substitute $u = x - 1$, $du = dx$:

$$I = \underbrace{\int_{u=-1}^{u=1} \frac{du}{\sqrt{1-u^2}}}_{\frac{1}{\sqrt{1-u^2}} \text{ is even!}} = 2\int_0^1 \frac{du}{\sqrt{1-u^2}} = 2\lim_{c\to 1^-} \int_0^c \frac{du}{\sqrt{1-u^2}}$$

$$= 2\lim_{c\to 1^-} \left(\sin^{-1} c - \sin^{-1} 0\right)$$

$$= 2\left(\frac{\pi}{2} - 0\right) = \pi. \quad \square$$

3. $\int_0^1 \ln x\,dx = \lim_{c\to 0^+} \int_c^1 \ln x\,dx = \lim_{c\to 0^+} (x \ln x - x)\Big|_c^1$

$$= \lim_{c\to 0^+} \Big((0-1) - (\underbrace{c \ln c}_{0\cdot\infty=\frac{\infty}{\infty}} - c)\Big)$$

$$= -1 - \lim_{c\to 0^+} \underbrace{\frac{\ln c}{\frac{1}{c}}}$$ —using L'Hopital's Rule!

$$= \lim_{c\to 0^+} \frac{(\ln c)'}{\left(\frac{1}{c}\right)'} = \lim_{c\to 0^+} \frac{\frac{1}{c}}{-\frac{1}{c^2}} = \lim_{c\to 0^+} (-c) = 0$$

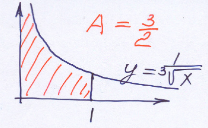
Answer: $\boxed{I = -1}$. □

4. $\displaystyle\int_0^a \frac{dx}{x^p} = \lim_{c\to 0^+} \int_c^a x^{-p}dx.$ (For $\underline{p=1}$, see Example 1.)

For $\underline{p \neq 0}$: $\displaystyle I = \lim_{c\to 0^+} \left.\frac{x^{-p+1}}{-p+1}\right|_c^a = \lim_{c\to 0^+} \frac{a^{-p+1}-c^{-p+1}}{-p+1},$

where $\displaystyle\lim_{c\to 0^+} c^{-p+1} = \begin{cases} 0, & \text{if } p<1 \\ +\infty, & \text{if } p>1 \end{cases} \Rightarrow \boxed{\int_0^a \frac{dx}{x^p} = \begin{cases} \frac{a^{-p+1}}{-p+1}, & \text{if } p<1 \\ \text{diverges}, & \text{if } p \geqslant 1 \end{cases}}$

$\underline{\text{E.g.:}}$ $\displaystyle\underbrace{\int_0^1 \frac{dx}{\sqrt[3]{x}}}_{p=\frac{1}{3}} = \frac{3}{2}$—finite:



But $\displaystyle\underbrace{\int_0^1 \frac{dx}{x\sqrt[3]{x}}}_{p=\frac{4}{3}}$ diverges,

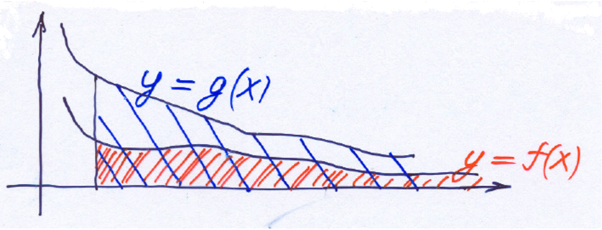
and the area A is negative for the curve $y = \frac{1}{x\sqrt[3]{x}}$.

§12.3 Comparison Test for Improper Integrals (Type I, II)

Comparison Test for Improper Integrals (both Type I, II)

Let $0 \leqslant f(x) \leqslant g(x)$. If $\int g(x)\,dx$ converges, so does $\int f(x)\,dx$.
If $\int f(x)\,dx$ diverges, so does $\int g(x)\,dx$.

Idea: $A_f \leqslant A_g$



So if $A_f = +\infty \Rightarrow$ so is $A_g = +\infty$; if A_g is finite $\Rightarrow$ so A_f is also finite.

Examples:

1. $\int_1^\infty e^{-x^2} dx$:

   For $x \geqslant 1$: $x^2 \geqslant x \quad \Rightarrow \; -x^2 \leqslant -x \quad \Rightarrow \; 0 < e^{-x^2} \leqslant e^{-x}$.

   So $0 < \int_1^\infty e^{-x^2} dx \leqslant \int_1^\infty e^{-x} dx = -e^{-x}\Big|_1^\infty = e^{-\infty} + e^{-1} = \frac{1}{e}$.

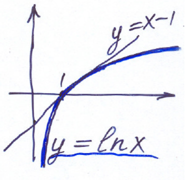
   So $\int_1^\infty e^{-x^2} dx$ exists (i.e. converges) and has its value in $(0, \frac{1}{e})$.
   □

2. $\displaystyle\int_2^\infty \frac{dx}{\ln x}$:

For $x > 0$: $\quad e^x = 1 + x + \frac{x^2}{2!} + \cdots > x \quad \Rightarrow \quad x > \ln x$;

So $\boxed{\dfrac{1}{x} < \dfrac{1}{\ln x}}$ NOTE also a sharper bound:

$$\frac{1}{x-1} < \frac{1}{\ln x}$$

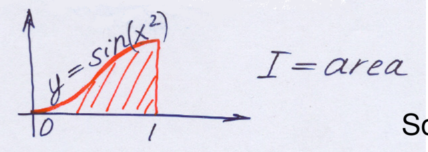


So $\int_2^\infty \frac{dx}{\ln x} > \int_2^\infty \frac{dx}{x} = \ln|x|\Big|_2^\infty = \ln(\infty) - \ln 2 = \infty.$

Hence $\int_2^\infty \frac{dx}{\ln x}$ diverges. □

Lecture 13 Numerical Integration

Integrals such as $\int_0^1 \sin(x^2)\,dx$ have no exact formulae, but yet exist as a certain area:



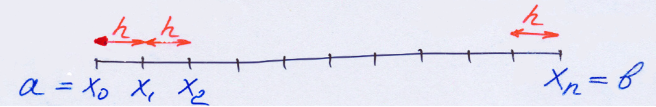
So one can evaluate them numerically...

§13.0 Notation

We want to approximate $\boxed{I = \int_a^b f(x)\,dx}$.

Assume: $f(x)$ is as many times differentiable as necessary on $[a, b]$.

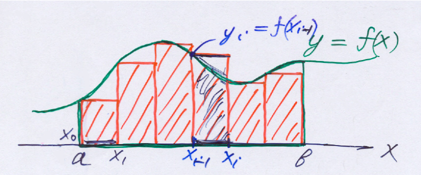
Partition $[a, b]$ into n equal subintervals of width h (earlier we used the notation $\triangle x$):



$$\boxed{x_i = a + ih, \quad i = 0, 1, 2, \cdots, n; \qquad h = \frac{b-a}{n}; \quad \text{define } y_i = f(x_i)}$$

§13.1 Rectangular Rule

On each subinterval $[x_{i-1}, x_i]$, approximate $f(x)$ by the leftmost value:



$$f(x) \approx f(x_{i-1}) = y_{i-1} \quad \text{on } [x_{i-1}, x_i]$$

$$\Rightarrow \quad \int_{x_{i-1}}^{x_i} f(x)\,dx \approx h\,y_{i-1}$$

$$\Rightarrow \quad \boxed{I \approx h\left(y_0 + y_1 + \cdots + y_{n-1}\right) = R_n}$$

(NOTE: this is just a left Riemann sum!)

Example: $I = \int_0^1 \sin(x^2)\,dx$. Use $h = \frac{1}{n}$, $x_i = ih = \frac{i}{n}$, $y_i = \sin\left(\frac{i^2}{n^2}\right)$

$$\boxed{I \approx R_n = \sum_{i=0}^{n-1} y_i\,h = \frac{1}{n}\sum_{i=0}^{n-1} \sin\left(\frac{i^2}{n^2}\right)}$$ yields:

n	10	100	1000	10 000	100 000
R_n	.269	.306	.3098	.310226	.3102641

Observe: Correct approximation: .3102683. So multiplying n by 10, roughly, gives one extra decimal place of accuracy!

Error Analysis

We want to estimate $\boxed{I - R_n}$ and be able to choose n so that the error $I - R_n$ is as small as we want.

The error in the Rectangular Rule

$$|I - R_n| \leqslant \frac{(b-a)^2}{2n} M_1, \quad \text{where } M_1 = \max_{x \in [a,b]} |f'(x)|.$$

NOTE: this theoretical error bound is very practical!
One typically does NOT know I, but still can estimate the error $|I - R_n|$ and choose an appropriate n before the computation occurs...

Application to our Example: $I = \int_0^1 \sin(x^2)\, dx$.

$$f'(x) = \left(\sin(x^2)\right)' = 2x \cdot \cos(x^2) \Rightarrow M_1 = \max_{x \in [0,1]} 2|x|\,|\cos(x^2)| \leqslant 2 \cdot 1 \cdot 1 = 2$$

$$\Rightarrow \boxed{\left|I - R_n\right| \leqslant \frac{(1-0)^2}{2n} \cdot 2 = \frac{1}{n}} \quad \text{E.g., if } n = 1000\text{: } \left|I - R_n\right| \leqslant \frac{1}{1000} = .001.$$

$$\text{In fact,} \quad \left|I - R_{1000}\right| = \left|.31026 - .30981\right| \approx .0004 < .001. \qquad \square$$

Proof of the Error Bound* (a bit technical):

First, we focus on the error associated with $[x_0, x_1]$.

NOTE: For an arbitrary function $g(x)$ one has:

$$\int_{x_0}^{x_1} g(x)\,dx = \underbrace{(x - x_1)\,g(x)\Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} (x - x_1)\,g'(x)\,dx}_{\text{here we used integration by parts with } u=g(x),\ v=(x-x_1)}$$

$$\Rightarrow\ \int_{x_0}^{x_1} g(x)\,dx = h\,g(x_0) - \int_{x_0}^{x_1} (x - x_1)\,g'(x)\,dx.$$

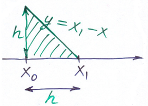
Next, choose $g(x) = f(x) - y_0$: then $g(x_0) = 0$, $g'(x) = f'(x)$, so

$$\Rightarrow\ \underbrace{\int_{x_0}^{x_1} \big(f(x) - y_0\big)\,dx}_{=\int_{x_0}^{x_1} f(x)dx - y_0 h} = 0 - \int_{x_0}^{x_1} (x - x_1)\,f'(x)\,dx$$

$$\Rightarrow\ \left|\int_{x_0}^{x_1} f(x)\,dx - y_0 h\right| = \left|\int_{x_0}^{x_1} (x - x_1)\,f'(x)\,dx\right| \leqslant \int_{x_0}^{x_1} \overbrace{\left|(x - x_1)\,f'(x)\right|}^{\leqslant (x_1 - x)\,M_1}\,dx$$

$$\Rightarrow \left|\int_{x_0}^{x_1} f(x)\,dx - y_0 h\right| = \left|\int_{x_0}^{x_1} (x - x_1)\, f'(x)\,dx\right| \leqslant \int_{x_0}^{x_1} \overbrace{\left|(x - x_1)\, f'(x)\right|}^{\leqslant (x_1 - x)\,M_1}\,dx$$

So $\left|\int_{x_0}^{x_1} f(x)\,dx - y_0 h\right| \leqslant M_1 \underbrace{\int_{x_0}^{x_1} (x_1 - x)\,dx}_{\text{here } \int_{x_0}^{x_1}(x_1 - x)\,dx = A = \frac{1}{2}h^2} \leqslant M_1 \cdot \frac{1}{2} h^2$—finally!



Similar estimates for each $[x_i, x_{i+1}]$: $\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - y_i h\right| \leqslant M_1 \cdot \frac{1}{2} h^2$.

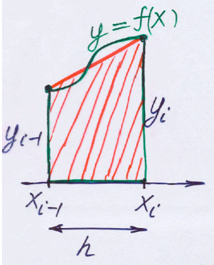
Now, compare $I = \int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx$ and $R_n = \sum_{i=0}^{n-1} h\,y_i$:

$$|I - R_n| = \left|\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}} f(x)\,dx - h\,y_i\right)\right| \leqslant \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - h\,y_i\right|$$

$$\leqslant \sum_{i=0}^{n-1}\left(M_1 \cdot \frac{1}{2}h^2\right) = M_1\,\frac{h^2}{2}\sum_{i=0}^{n-1} 1 = M_1\,\frac{h^2}{2}\,n = M_1\,\frac{(b-a)^2}{2n^2}\,n = M_1\,\frac{(b-a)^2}{2n}. \qquad \square$$

§13.2 Trapezoidal Rule

Approximate $y = f(x)$ on each $[x_{i-1}, x_i]$ by the straight line segment joining (x_{i-1}, y_{i-1}) and (x_i, y_i):



$$\int_{x_{i-1}}^{x_i} f(x)\,dx \approx A = h \cdot \frac{y_{i-1}+y_i}{2}$$

So we approximate:

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$\approx \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \cdots + \frac{h}{2}(y_{n-1} + y_n)$$

$$= \boxed{\frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) = T_n}$$

Error Estimate for the Trapezoidal Rule

$$|I - T_n| \leqslant \frac{(b-a)^3}{12n^2} M_2, \quad \text{where } M_2 = \max_{x\in[a,b]} |f''(x)|.$$

Proof: similar to $|I - R_n|$....

Example: $I = \int_0^1 \sin(x^2)\,dx$. Use $h = \frac{1}{n}$, $x_i = ih = \frac{i}{n}$, $y_i = \sin\left(\frac{i^2}{n^2}\right)$

$$I \approx \boxed{T_n = \frac{1}{2n}\left(0 + 2\sin\left(\frac{1}{n^2}\right) + 2\sin\left(\frac{2^2}{n^2}\right) + \cdots + 2\sin\left(\frac{(n-1)^2}{n^2}\right) + \sin(1^2)\right)}$$

n	10	100	1000	10 000
T_n	.3111	.310277	.31026839	.3102683026

Observe: Multiplying n by 10, roughly, yields 2-decimal-place increase in accuracy! This reflects the fact that $|I - T_n| \sim \frac{1}{n^2}$.

. . . .

Error Estimate: $f(x) = \sin(x^2) \;\Rightarrow\; f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$

$$\Rightarrow\; M_2 = \max_{x\in[0,1]} |f''(x)| \leqslant \max_{x\in[0,1]} \left(2\,|\cos(x^2)| + 4x^2|\sin(x^2)|\right)$$

$$\leqslant 2\cdot 1 + 4\cdot 1\cdot 1 = 6.$$

$$\Rightarrow\; \boxed{|I - T_n| \leqslant \frac{(1-0)^3}{12n^2}6 = \frac{1}{2n^2}} \quad \text{(Compare with } |I - R_n| \leqslant \tfrac{1}{n}!\text{)}$$

So if we want $|I - T_n| \leqslant 10^{-8}$, choose n such that $\frac{1}{2n^2} \leqslant 10^{-8}$

$$\Rightarrow\; 2n^2 \geqslant 10^8 \qquad \Rightarrow\; n \geqslant \sqrt{\frac{10^8}{2}} \approx 7071.06$$

So $n = 7072$ subintervals guarantee $|I - T_n| \leqslant 10^{-8}$. □

Problem: How many subintervals are needed to estimate

$I = \int_0^1 \sin(x^2)\,dx$ with error $\leqslant 10^{-8}$ using

(a) Trapezoidal Rule; (b) Rectangular Rule??

(a)—already done:

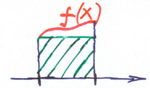
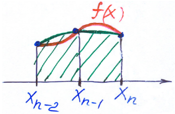
Recall that $|I - T_n| \leqslant \frac{1}{2n^2}$.

To guarantee $|I - T_n| \leqslant 10^{-8}$, choose $\frac{1}{2n^2} \leqslant 10^{-8}$ so $\boxed{n \geqslant 7072}$. □

(b) Recall that $|I - R_n| \leqslant \frac{1}{n}$ (see §13.1).

We want $|I - R_n| \leqslant 10^{-8}$, so choose $\frac{1}{n} \leqslant 10^{-8}$ so $\boxed{n \geqslant 10^8}$. □

Lecture 14 Numerical Integration: Simpson's Rule

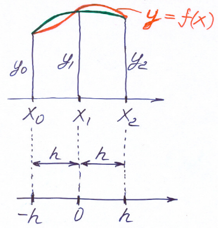
	On each subinterval $f(x)$ is approximated by:
Rectangular Rule	a constant function: 
Trapezoidal Rule	a linear function 
Simpson's Rule	a quadratic function  —to specify a quadratic function, we need 3 points, i.e. a pair of subintervals

As a pair of subintervals is needed $\Rightarrow$ Change of Notation:

partition $[a, b]$ into $2n$ subintervals

with $\boxed{h = \frac{b-a}{2n}}$

Consider $[x_0, x_2]$:

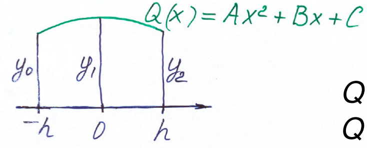


We approximate $\int_{x_0}^{x_2} f(x)\,dx$

by the area under the (green) quadratic curve

through $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

By "translating" this parallel to the x-axis, this is the area under the quadratic curve through $(-h, y_0)$, $(0, y_1)$ and (h, y_2).



What are A, B, C?

$$\left.\begin{aligned} Q(0) &= y_1 \\ Q(h) &= y_2 \\ Q(-h) &= y_0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} C &= y_1 \\ Ah^2 + Bh + C &= y_2 \\ Ah^2 - Bh + C &= y_0 \end{aligned}\right.$$

Add the last 2 relations: $2Ah^2 + 0 + 2C = y_0 + y_2 \Rightarrow \boxed{Ah^2 = \frac{y_0 - 2y_1 + y_2}{2}}$

What is the area under $y = Q(x)$??

$$\int_{x_0}^{x_2} f(x)\,dx \approx \int_{-h}^{h} Q(x)\,dx = \int_{-h}^{h} (Ax^2 + Bx + C)\,dx = \frac{2Ah^3}{3} + 0 + 2Ch$$

$$= \frac{2h}{3} \cdot \frac{y_0 - 2y_1 + y_2}{2} + 2\,y_1\,h = \frac{h}{3}(y_0 - 2y_1 + y_2 + 6y_1)$$

$$\Rightarrow \boxed{\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2)}$$

Repeating this for each pair of subintervals yields:

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{2n-2}}^{x_{2n}} f(x)\,dx$$

$$\approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{2n-2} + 4y_{2n-1} + y_{2n})$$

Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{2n-2} + 4y_{2n-1} + y_{2n})$$

$$= S_{2n} = \frac{h}{3}\left(y_0 + 4\sum y_{\text{"odds"}} + 2\sum y_{\text{"evens"}} + y_{2n}\right)$$

Error Estimate for Simpson's Rule

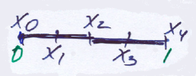
$|I - S_{2n}| \leqslant \frac{h^4}{180}(b-a)\,M_4 = \frac{(b-a)^5}{180\,(2n)^4}\,M_4$, where $M_4 = \max_{x\in[a,b]} |f^{(iv)}(x)|$.

(without proof)

—the error decreases as $\frac{1}{(2n)^4}$. So 10-fold increase in n yields a 10 000-fold increase in accuracy!

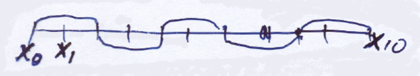
Example: $I = \int_0^1 \sin(x^2)\,dx \approx S_{2n}$

- Let $2n = 4$: $\Rightarrow$ $I \approx S_4 = \frac{\left(\frac{1}{4}\right)}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$



where $x_i = \frac{i}{2n} = \frac{i}{4}$ so $y_i = \sin\left(\frac{i^2}{(2n)^2}\right) = \sin\left(\frac{i^2}{16}\right)$.

- Now let $2n = 10$:



$$I \approx S_{10} = \frac{\left(\frac{1}{10}\right)}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_8 + 4y_9 + y_{10})$$

where $x_i = \frac{i}{2n} = \frac{i}{10}$ so $y_i = \sin\left(\frac{i^2}{(2n)^2}\right) = \sin\left(\frac{i^2}{10^2}\right)$.

$2n$	S_{2n}	correct decimal places
4	.3099	2
10	.31026023	5
100	.31026830092	9
1000	.310268301723301	13

Problem: How many subintervals are needed to estimate

$I = \int_0^1 \sin(x^2)\,dx$ with error $\leqslant 10^{-8}$ using Simpson's Rule?

S: Recall that $|I - S_{2n}| \leqslant \frac{(b-a)^5}{180\,(2n)^4} M_4$.

$$f^{(iv)}(x) = \frac{d^4}{dx^4}\left(\sin(x^2)\right) = (16x^4 - 12)\,\sin(x^2) - 48x^2\,\cos(x^2).$$

Note: for $x \in [0,1]$ one has $|16x^4 - 12| \leqslant 12$ and $x^2 \leqslant 1$so

$$\Rightarrow M_4 = \max_{x\in[0,1]} \left|\frac{d^4}{dx^4}\left(\sin(x^2)\right)\right| \leqslant 12 \cdot 1 + 48 \cdot 1 \cdot 1 = 60.$$

$$\Rightarrow |I - S_{2n}| \leqslant \frac{(1-0)^5}{180\,(2n)^4} 60 \leqslant \frac{1}{3\,(2n)^4}.$$

To guarantee $|I - S_{2n}| \leqslant 10^{-8}$, choose $\boxed{\frac{1}{3\,(2n)^4} \leqslant 10^{-8}}$

so $2n \geqslant \left(\frac{10^8}{3}\right)^{\frac{1}{4}} = \frac{100}{\sqrt[4]{3}} \approx 76.$

Answer: $\boxed{2n \geqslant 76}$ will suffice. □

Remark: for polynomials of degree $\leqslant 3$ one has

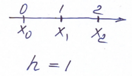
$$f^{(iv)}(x) = 0 \Rightarrow M_4 = 0 \Rightarrow |I - S_{2n}| = 0,$$

i.e., Simpson's Rule is exact: $\boxed{S_{2n} = I}$ for any $2n$!

Example:

Evaluate $\int_0^2 x^3 dx$ using Simpson's Rule with $n = 1$ (and $2n = 2$).

S:

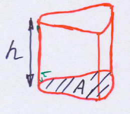


$$y_0 = 0^3 = 0,\ y_1 = 1^3 = 1,\ y_2 = 2^3 = 8,$$

$$\Rightarrow S_2 = \frac{1}{3}\left(y_0 + 4y_1 + y_2\right) = \frac{1}{3}\left(0 + 4 \cdot 1 + 8\right) = 4$$

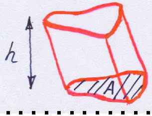
—exact answer! □

Note that the volume of a general cylinder



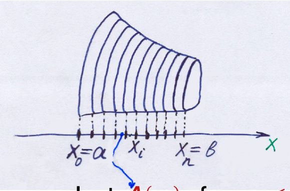
$\boxed{V = A \cdot h}$, where
A is the area of the base,
and h is its height.

The same formula $V = A \cdot h$ is valid an oblique cylinder:



Note:
here h is measured in the direction perpendicular to the base!

Consider a general solid:



—Divide it into thin slices
by parallel planes perpendicular to an axis.

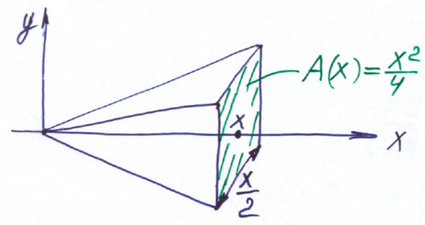
Let $A(x)$, for $a \leqslant x \leqslant b$ be the cross-sectional area.

Each thin slice is approximately a cylinder of height $\triangle x_i$ with base area $A(x_i)$. So its volume is $\boxed{\triangle V_i \approx A(x_i)\cdot\triangle x_i}$

The volume of the solid: $V = \sum_{i=1}^{n} \triangle V_i \approx \underbrace{\sum_{i=1}^{n} A(x_i)\cdot\triangle x_i}_{\text{Riemann sum!}}$

Let $n\to\infty$: $\boxed{V = \int_a^b A(x)\,dx}$ where $A(x)$ is the cross-sectional area.

Example (volume of a pyramid)



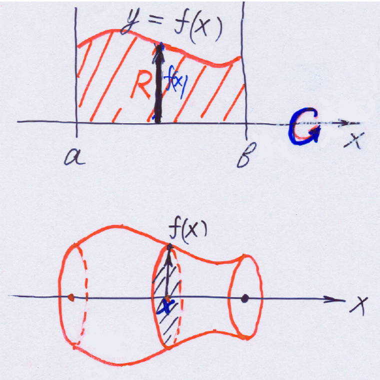
Suppose that a pyramid has cross-sectional area $A(x) = \frac{x^2}{4}$ for $0 \leqslant x \leqslant 4$.

Then $V = \int_0^4 A(x)\,dx = \int_0^4 \frac{x^2}{4}\,dx$

$= \frac{x^3}{12}\Big|_0^4 = \frac{16}{3}$. □

§15.2 Solids of Revolution: Cross-Section is a Disc

If the region R bounded by $y = f(x) \geqslant 0, \quad y = 0, \quad x = a, \quad x = b,$



is rotated
about the x-axis,

then the cross-section at x
is a circular disk of radius $f(x)$,

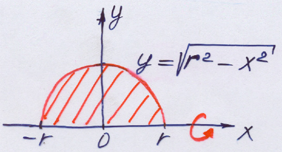
whose area $\boxed{A(x) = \pi\left(f(x)\right)^2}$

$$\Rightarrow \quad \boxed{V = \int_a^b A(x)\,dx = \pi \int_a^b \left(f(x)\right)^2 dx}$$

Examples:

1. Find the volume of a ball of radius r.

S:



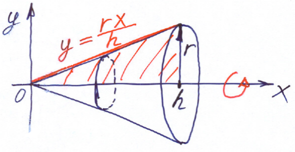
The ball can be generated by rotating the half-disc about the x-axis

$$\Rightarrow\ V=\pi\int_{-r}^{r}\big(f(x)\big)^2dx=\pi\int_{-r}^{r}\underbrace{(r^2-x^2)}_{\text{even function}}dx$$

$$=2\pi\int_0^r(r^2-x^2)\,dx=2\pi\left(r^2x-\frac{x^3}{3}\right)\Big|_{x=0}^{x=r}=2\pi\left(\left(r^3-\frac{r^3}{3}\right)-0\right)=\frac{4\pi}{3}r^3.\ \square$$

2. A right circular cone of radius r and height h:

S: It's generated by rotating the line $y = \frac{r\,x}{h}$ about the x-axis for $0 \leqslant x \leqslant h$.

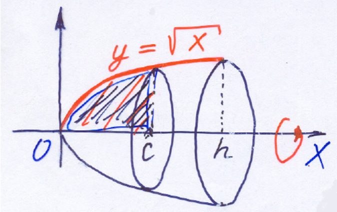


$$\Rightarrow \quad V = \pi \int_0^h \big(f(x)\big)^2 dx$$

$$= \pi \int_0^h \left(\frac{r\,x}{h}\right)^2 dx$$

$$= \pi \frac{r^2}{h^2} \int_0^h x^2 dx = \pi \frac{r^2}{h^2} \frac{x^3}{3}\Big|_{x=0}^{x=h} = \pi \frac{r^2}{h^2} \frac{h^3}{3} - 0 = \frac{\pi\, r^2 h}{3}. \qquad \square$$

3. A wine glass is designed by rotating $y = \sqrt{x}$, for $0 \leqslant x \leqslant h$ about the x-axis.
What is the depth of wine in the glass when it is half-full?

S: 

When the depth of wine in the glass is c, the volume of wine is

$$V(c) = \pi \int_0^c \left(\sqrt{x}\right)^2 dx = \pi \int_0^c x\, dx = \pi \left.\frac{x^2}{2}\right|_0^c = \frac{\pi c^2}{2}.$$

$\Rightarrow$ $V(h) = \frac{\pi h^2}{2}$ is the volume when the glass is full.

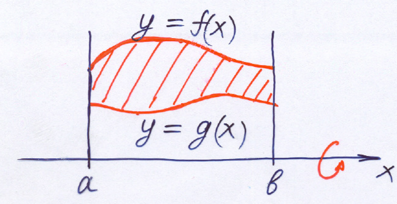
The glass is half-full when c is such that

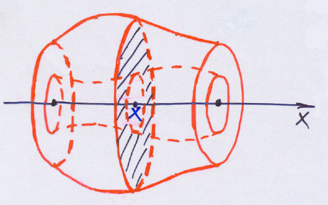
$$\boxed{V(c) = \frac{1}{2} V(h)} \quad \Rightarrow \quad \frac{\pi c^2}{2} = \frac{1}{2} \frac{\pi h^2}{2} \quad \Rightarrow \quad c^2 = \frac{1}{2} h^2$$

$\Rightarrow$ $\boxed{c = \frac{1}{\sqrt{2}} h \approx .7071\, h}$. Answer: the depth is 70% of total. □

§15.3 Solids of Revolution: Cross-Section is a Washer

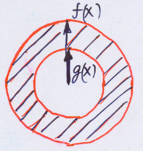
If the region R bounded by $y = f(x)$, $y = g(x)$, $x = a$, $x = b$,





is rotated
about the x-axis,

then the cross-section at x
is a washer of outer radius $f(x)$,
inner radius $g(x)$:



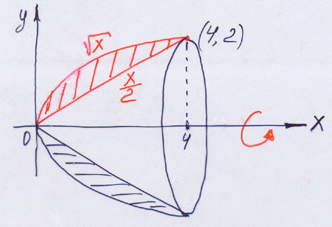
Hence the area of the cross-section at x:

$$A(x) = \underbrace{\pi\left(f(x)\right)^2}_{\text{outer disk}} - \underbrace{\pi\left(g(x)\right)^2}_{\text{inner disk}}$$

$$\Rightarrow \quad \boxed{V = \int_a^b A(x)\,dx = \pi \int_a^b \left(\left(f(x)\right)^2 - \left(g(x)\right)^2\right) dx}$$

Examples:

1. The area between $y = \sqrt{x}$ and $y = \frac{x}{2}$ is rotated about the x-axis. Find the volume generated.



$$V = \pi \int_0^4 \left(\left(\sqrt{x}\right)^2 - \left(\tfrac{x}{2}\right)^2 \right) dx$$

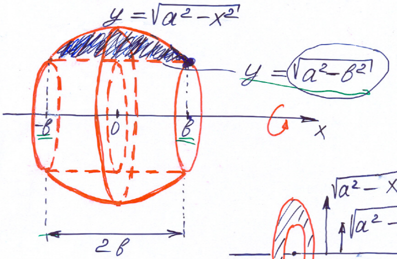
$$= \pi \int_0^4 \left(x - \tfrac{x^2}{4}\right) dx$$

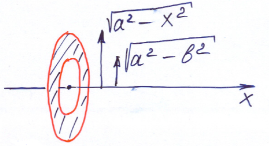
$$= \pi \left(\tfrac{x^2}{2} - \tfrac{x^3}{12}\right)\Big|_0^4 = \pi \left(\tfrac{4^2}{2} - \tfrac{4^3}{12}\right) - 0 = \tfrac{8\pi}{3}. \qquad \square$$

2. A cylindrical hole of height $2b$ is drilled thought the centre of a ball. Find the remaining volume.

S: Let a be the radius of the ball.

The solid can be generated by rotating the region:





The cross-section at each $x \in [-b, b]$ is a washer

$$\Rightarrow V = \pi \int_{-b}^{b} \left(\left(\sqrt{a^2 - x^2}\right)^2 - \left(\sqrt{a^2 - b^2}\right)^2 \right) dx$$

$$= \pi \int_{-b}^{b} \left((\underline{a^2} - x^2) - (\underline{a^2} - b^2) \right) dx$$

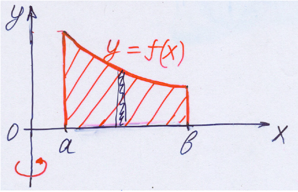
$$= \pi \int_{-b}^{b} \underbrace{(b^2 - x^2)}_{\text{even}} dx = 2\pi \int_{0}^{b} (b^2 - x^2)\, dx = 2\pi \left(b^2 x - \frac{x^3}{3} \right) \Big|_{x=0}^{x=b}$$

$$= 2\pi \left(b^3 - \frac{b^3}{3} \right) - 0 = \frac{4\pi b^3}{3}. \qquad \square$$

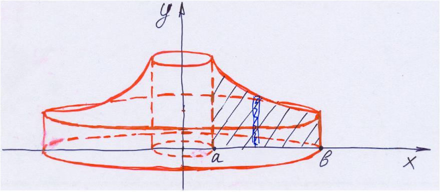
Note: the answer is independent of a!

Lecture 16 §16.1 Volumes of Revolution: by Cylindrical Shells

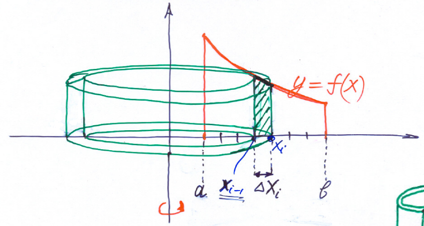
Suppose the region bounded by $y = f(x) \geqslant 0, \quad y = 0, \quad x = a, \quad x = b,$



is rotated about the y-axis:

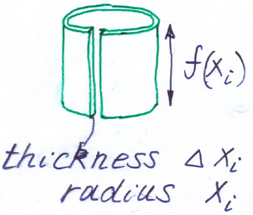


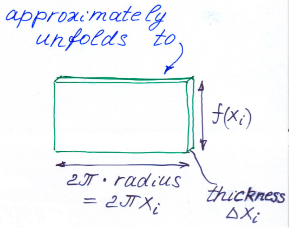
- One approach: by slicing perpendicular to the y-axis (see Lecture 15).Then at each y, we need the cross-sectional area $A(y)$ —sometimes NOT easy to obtain...
- Alternatively, (we shall employ this!) divide the solid into thin cylindrical shells $\Rightarrow$



–First, divide the region into thin vertical strips.

–Then rotate each strip about the y-axis to generate a thin cylindrical shell:





⇒ The volume of a particular shell: $\boxed{\triangle V_i \approx 2\pi\, x_i\, f(x_i)\, \triangle x_i}$

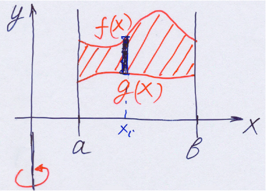
⇒ The volume of the whole solid: $V = \sum_{i=1}^{n} \triangle V_i \approx 2\pi \underbrace{\sum_{i=1}^{n} x_i\, f(x_i)\, \triangle x_i}_{\text{Riemann Sum}}$

(where n is the number of vertical strips).

Let $n \to \infty \Rightarrow \boxed{V = 2\pi \int_a^b x\, f(x)\, dx}$

Generalization:

If the region bounded by $y = f(x)$, $y = g(x)$, $x = a$, $x = b$,



is rotated

about the y-axis, then the volume is

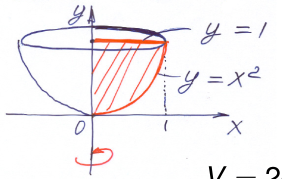
$$V = 2\pi \int_a^b x\,(f(x) - g(x))\,dx \quad (*)$$

Hint: the vertical strip at x_i has height $f(x_i) - g(x_i)$

(earlier it was $f(x_i)$)...

Examples:

1. Find the volume of the bowl generated by rotating $y = x^2$ about the y-axis for $0 \leqslant x \leqslant 1$.



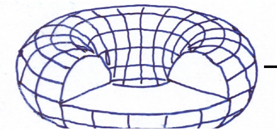
Use formula $(*)$

with the upper function $f(x) = 1$

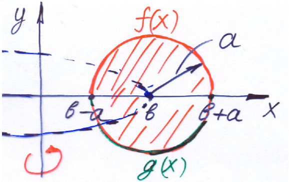
and the lower function $g(x) = x^2$:

$$V = 2\pi \int_0^1 x\,(1 - x^2)\,dx = 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}. \quad \square$$

2. Volume of a Torus (Donut):



—cutaway view of a torus.



—A disc of radius a
with centre $(b, 0)$, where $b > a$
is rotated about the y-axis

By formula $(*)$: $V = 2\pi \int_{b-a}^{b+a} x \left(f(x) - g(x)\right) dx$, but

What are $f(x)$ and $g(x)$?

The circle: $(x-b)^2 + (y-0)^2 = a^2 \;\Rightarrow\; y = \pm\sqrt{a^2-(x-b)^2}$

upper branch: $+\sqrt{a^2-(x-b)^2} = f(x)$

lower branch: $-\sqrt{a^2-(x-b)^2} = g(x)$

Hence, $V = 2\pi \int_{b-a}^{b+a} x \Big(\underbrace{\sqrt{a^2-(x-b)^2}}_{f(x)} - \underbrace{\left(-\sqrt{a^2-(x-b)^2}\right)}_{g(x)}\Big)\, dx$

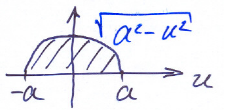
So $V = 4\pi \int_{b-a}^{b+a} x\sqrt{a^2-(x-b)^2}\,dx$

Substitute: $u = x - b \quad \Rightarrow \quad du = dx$
with limits: $x = b + a \quad \Rightarrow \quad u = a, \quad x = b - a \quad \Rightarrow \quad u = -a$:

$$\Rightarrow \quad V = 4\pi \int_{u=-a}^{u=a} (u+b)\sqrt{a^2-u^2}\,du$$

$$= 4\pi \underbrace{\int_{-a}^{a} u\sqrt{a^2-u^2}\,du}_{=0,\text{ as odd function}} + 4\pi b \underbrace{\int_{-a}^{a} \sqrt{a^2-u^2}\,du}_{\text{area of semicircle of radius } a:}$$

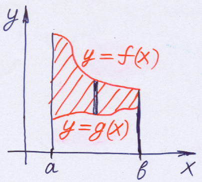
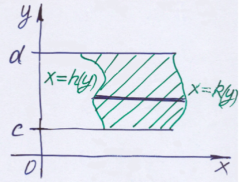
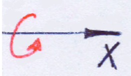
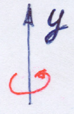
area $= \frac{\pi a^2}{2}$



$$= 0 + 4\pi b \cdot \frac{\pi a^2}{2} = \boxed{(2\pi b)(\pi a^2)} \qquad \square$$

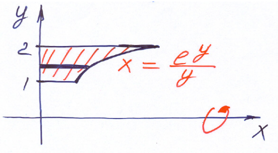
Note:
area of the disc (πa^2) times the distance traveled by its centre $(2\pi b)$.

§16.2 Volumes of Revolution: Summary and Generalizations

Region→ Rotated about ↓		
x-axis 	plane vertical slices $V = \pi \int_a^b \left((f(x))^2 - (g(x))^2\right) dx$	horizontal cylindrical shells $V = 2\pi \int_c^d y\,\left(k(y) - h(y)\right)\, dy$
y-axis 	vertical cylindrical shells $V = 2\pi \int_a^b x\,\left(f(x) - g(x)\right)\, dx$	plane horizontal slices $V = \pi \int_c^d \left((k(y))^2 - (h(y))^2\right) dy$
	—See Lectures 15–16	—This column is new and generalizes the previous one!

Examples:

1. The area bounded by $x = \frac{e^y}{y}$ and the y-axis for $1 \leqslant y \leqslant 2$:



is rotated about the x-axis.

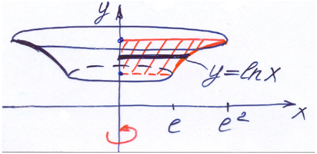
Find the volume generated.

S: Use horizontal cylindrical shells:

$$V = 2\pi \int_{y=1}^{y=2} y \Big(\underbrace{\frac{e^y}{y}}_{\text{"right" curve}} - \underbrace{0}_{\text{"left" curve: } x-\text{axis}} \Big)\, dy = 2\pi \cdot e^y \Big|_{y=1}^{y=2} = 2\pi\,(e^2 - e).$$

□

2. Find the volume generated by rotating $y = \ln x$ about the y-axis for $e \leqslant x \leqslant e^2$.



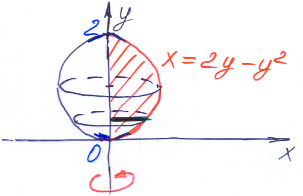
Use horizontal slices:

$$V = \pi \int_{y=??}^{y=??} \Big((\underbrace{\text{right}}_{y=\ln x,\text{ but } x=e^y \text{ as a function of } y})^2 - (\underbrace{\text{left}}_{y-\text{axis: } x=0})^2 \Big)\, dy$$

Limits: $x = e \;\Rightarrow\; y = \ln e = 1$, and $x = e^2 \;\Rightarrow\; y = \ln e^2 = 2$

$$V = \pi \int_{y=1}^{y=2} \big((e^y)^2 - 0^2 \big)\, dy = \pi \int_1^2 e^{2y}\, dy = \frac{\pi}{2} e^{2y} \Big|_1^2 = \frac{\pi}{2} \left(e^4 - e^2 \right). \quad \square$$

3. Find the volume generated by rotating the region bounded by $x = 2y - y^2$ and the y-axis about the y-axis.



Intersections: $x = 2y - y^2$ and $x = 0$

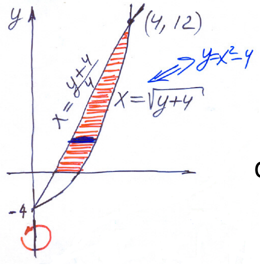
$$\Rightarrow \; 2y - y^2 = 0 \text{ so } \boxed{y = 0, 2}.$$

Horizontal slices:

$$\Rightarrow \; V = \pi \int_{y=0}^{y=2} \Big((\underbrace{2y - y^2}_{\text{right}})^2 - (\underbrace{\;0\;}_{\text{left: } y \text{ axis } x=0})^2 \Big)\, dy$$

$$= \pi \int_0^2 (4y^2 - 4y^3 + y^4)\, dy = \pi \left(4\,\frac{y^3}{3} - 4\,\frac{y^4}{4} + \frac{y^5}{5}\right)\Big|_0^2 = \frac{16\pi}{15}. \quad \square$$

4. The region bounded by $x = \frac{y+4}{4}$ and the $x = \sqrt{y+4}$ for $0 \leqslant y \leqslant 12$, is rotated about the y-axis.



Horizontal slices:

outer: $\sqrt{y+4}$; inner: $\frac{y+4}{4}$; limits: $y = 0,\ 12$.

$$\Rightarrow \quad V = \pi \int_{y=0}^{y=12} \left(\left(\sqrt{y+4}\right)^2 - \left(\tfrac{y+4}{4}\right)^2 \right) dy$$

Substitution: $u = \frac{y+4}{4} \;\Rightarrow\; du = \frac{dy}{4} \;\Rightarrow\; dy = 4du$,

with limits: $y = 0 \;\Rightarrow\; u = 1$, $y = 12 \;\Rightarrow\; u = 4$,

$$\Rightarrow \quad V = \pi \int_{u=1}^{u=4} (4u - u^2)\,(4du) = 4\pi \left(4\,\tfrac{u^2}{2} - \tfrac{u^3}{3}\right)\Big|_1^4 = 36\pi. \qquad \square$$