UNIVERSITY *of* LIMERICK

OLLSCOIL LUIMNIGH

College of Informatics and Electronics

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4008 SEMESTER: Autumn 2005

MODULE TITLE: Numerical Partial Differential Equations DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Dr. N. Kopteva PERCENTAGE OF TOTAL MARKS: 75%

INSTRUCTIONS TO CANDIDATES:
Answer questions 1, 2, and 3.

To obtain maximum marks you must show all your work clearly and in detail.
Standard mathematical tables are provided by the invigilators. Under no circumstances should you use your own tables or be in possession of any written material other than that provided by the invigilators.
Non-programmable, non-graphical calculators that have been approved by the lecturer are permitted.
You must obey the examination rules of the University. Any breaches of these rules (and in particular any attempt at cheating) will result in disciplinary proceedings. For a first offence this can result in a year's suspension from the University.

1 Answer part (a) and one of parts (b) and (c). **30%**

. . . .

(a) Let V be a Hilbert space, $a(\cdot,\cdot)$ be a symmetric bilinear form, and $L(\cdot)$ be a linear form on V. Furthermore, let $a(v,v)\geq 0 \quad \forall v\in V$. 15%

Consider the following problems:

- *Minimization problem (MIN)*:
  Find $u\in V$ such that $F(u)=\min\limits_{v} F(v)$, where $F(v)=\frac{1}{2}a(v,v)-L(v)$.
- *Variational problem (VAR)*:
  Find $u\in V$ such that $a(u,v)=L(v) \quad \forall v\in V$.

Prove the following statement:

If u is a solution of problem (VAR), then u is a solution of problem (MIN).

. . . .

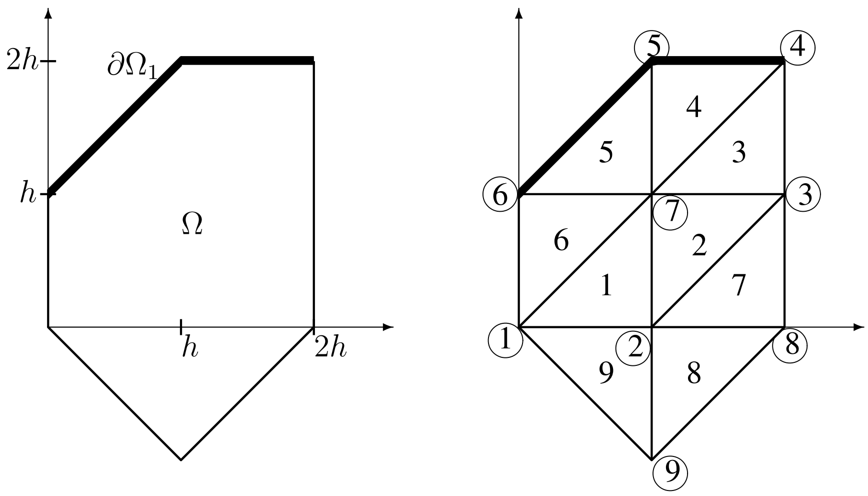
(b) In a two-dimensional domain Ω consider the problem: 15%

$$-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f,\quad (x,y)\in\Omega,$$

$$u(x,y)=0,\quad (x,y)\in\partial\Omega_1;\qquad \frac{\partial u(x,y)}{\partial \mathbf{n}}=0,\quad (x,y)\in\partial\Omega_2;$$

where $f=\text{const}$ and $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$ is the boundary of Ω.

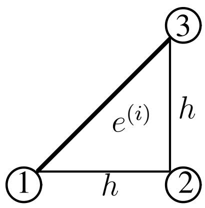
This problem is discretized using *linear finite elements* on the following triangulation of the following domain Ω :



For this discretization:

- Find the global stiffness matrix $K_{(f)}$ and the global load vector $F_{(f)}$ in which the boundary conditions are ignored.
- Find the global stiffness matrix K and the global load vector F that take the boundary conditions into consideration.
- Then write the numerical method as a linear system $KU = F$. For each entry of the unknown vector U specify to which mesh node it is assigned.

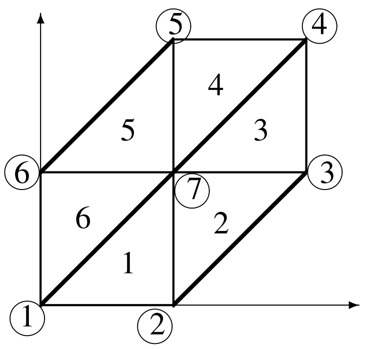
Note that for the linear element



the local stiffness matrix $K^{(i)}$ and the local load vector $F^{(i)}$ are given by

$$K^{(i)} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad F^{(i)} = \frac{fh^2}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Note also that for the triangulation



we have

$$K_{(f)} = \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 3 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 3 & -1 & 0 & 0 & -2 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & 0 & -2 \\ -1 & 0 & 0 & 0 & 0 & 3 & -2 \\ 0 & -2 & -2 & 0 & -2 & -2 & 8 \end{bmatrix}, \quad F_{(f)} = \frac{fh^2}{6}\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 6 \end{bmatrix}.$$

. . . .

(c) Consider the problem: 15%

$$-u'' + 3u' = f, \quad x \in (0,1), \qquad u(0) = u(1) = 0.$$

- Obtain a weak formulation of this problem.
(Note: you are expected to specify the space in which u is found and from which arbitrary functions v are taken.)
- Suppose this problem is discretized using piecewise quadratic finite elements with the local shape functions $\phi_1^{(i)}$, $\phi_2^{(i)}$, $\phi_3^{(i)}$ defined on each element $e^{(i)} = [x_{i-1}, x_i]$ by

$$\phi_k^{(i)} = \varphi_k\left(\frac{x - x_{i-1}}{h_i}\right), \quad k = 1, 2, 3,$$

where $h_i = x_i - x_{i-1}$ and

$$\varphi_1(t) = 2(t-1)(t-1/2), \quad \varphi_2(t) = 4t(1-t), \quad \varphi_3(t) = 2t(t-1/2).$$

Find the local stiffness matrix $K^{(i)}$ and the local load vector $F^{(i)}$, assuming that $f = \mathrm{const}$.

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2 Answer part (a) and any *two* of parts (b), (c), (d). 20%

In the two-dimensional domain $\Omega = (0,1) \times (0,1)$ with the boundary $\partial\Omega$ consider the problem:

$$Lu \equiv -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2u = f(x,y), \quad (x,y) \in \Omega,$$

$$u(x,y) = 0, \quad (x,y) \in \partial\Omega.$$

This problem is discretized on the uniform mesh $\{(ih, jh), i, j = 0, \ldots, N\}$, where $h = 1/N$, by the finite difference method:

$$L^h U_{ij} \equiv -\frac{U_{i-1,j} + U_{i+1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}}{h^2} + 2U_{ij} = f(ih, jh),$$

for $i, j = 1, \ldots, N-1$, with the boundary conditions:

$$U_{ij} = 0, \qquad (ih, jh) \in \partial\Omega.$$

. . . .

(a) Estimate the local truncation error $L^h u_{ij} - f(ih, jh)$ of this method, where $u_{ij} = u(ih, jh)$ is the exact solution at the mesh node (ih, jh). 6%

. . . .

(b) Show that the finite difference operator L^h satisfies the maximum principle of the form: 7%

$$\left.\begin{array}{l} L^h v_{ij} \geq 0 \quad \forall i, j \\ v_{ij} \geq 0 \quad \forall (ih, jh) \in \partial\Omega \end{array}\right\} \quad \Rightarrow \quad v_{ij} \geq 0 \quad \forall\, i, j.$$

. . . .

(c) Using the maximum principle described in part (b), show that 7%

$$v_{ij} = 0 \text{ for } (ih, jh) \in \partial\Omega \quad \Rightarrow \quad \max_{i,j} \left| v_{ij} \right| \leq C \max_{i,j} \left| L^h v_{ij} \right|$$

for some constant C. Specify this constant.

. . . .

(d) Using the result of part (a) and the property described in part (c), estimate the error of the finite difference method 7%

$$\max_{i,j} \left| U_{ij} - u(ih, jh) \right|,$$

where U_{ij} is the computed solution, while $u(ih, jh)$ is the exact solution at the mesh node (ih, jh).

3 Answer part (a) and one of parts (b) and (c). 25%

Consider the problem

$$u_t = u_{xx}, \;\; x \in (0,1), \; t > 0, \quad u(0,t) = u(1,t) = 0, \quad u(x,0) = u_0(x).$$

This problem is discretized on the uniform mesh $\{(x_j, t^m), \; j = 0, \ldots, N,$ $m = 0, 1, \ldots\}$, where $x_j = jh$ and $t^m = mk$.
Furthermore, let U_j^m be the computed solution at the mesh node (x_j, t^m).

. . . .

(a) Using Von Neumann's method, prove that the backwards differentiation method 13%

$$\frac{1}{k}\left(\frac{3}{2}\,U_j^{m+1} - 2\,U_j^m + \frac{1}{2}\,U_j^{m-1}\right) = \frac{U_{j-1}^{m+1} - 2U_j^{m+1} + U_{j+1}^{m+1}}{h^2},$$

is *unconditionally stable*.

. . . .

(b) Find the local truncation error of the method in part (a). 12%

. . . .

(c) Find the local truncation error of the following method: 12%

$$\frac{U_j^{m+1} - U_j^m}{k} = \frac{1}{4}\,\frac{U_{j-1}^{m+1} - 2U_j^{m+1} + U_{j+1}^{m+1}}{h^2} + \frac{3}{4}\,\frac{U_{j-1}^{m} - 2U_j^{m} + U_{j+1}^{m}}{h^2}.$$

Furthermore, using Von Neumann's method, find out whether this method is unconditionally stable, unconditionally unstable, or conditionally stable. If it is conditionally stable, find the stability condition.