UNIVERSITY *of* LIMERICK
OLLSCOIL LUIMNIGH

FACULTY OF SCIENCE AND ENGINEERING

DEPARTMENT OF MATHEMATICS & STATISTICS

END OF SEMESTER ASSESSMENT PAPER

MODULE CODE: MS4008 SEMESTER: Autumn 2015/16

MODULE TITLE: Numerical Partial Differential Equations DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

LECTURER: Prof. N. Kopteva PERCENTAGE OF TOTAL MARKS: 75%

EXTERNAL EXAMINER: Prof. J. King

INSTRUCTIONS TO CANDIDATES:
Answer questions 1, 2, and 3.

To obtain maximum marks you must show all your work clearly and in detail.
Standard mathematical tables are provided by the invigilators. Under no circumstances should you use your own tables or be in possession of any written material other than that provided by the invigilators.
Non-programmable, non-graphical calculators that have been approved by the lecturer are permitted.
You must obey the examination rules of the University. Any breaches of these rules (and in particular any attempt at cheating) will result in disciplinary proceedings. For a first offence this can result in a year's suspension from the University.

1 Answer part (a) and one of parts (b) and (c). **30%**

. . . .

(a) Let the space $V = \big\{\text{all functions } v \in H^1(0,1) \text{ such that } v(1) = 0\big\}$ with the norm 15%

$$\|v\| = \sqrt{\int_0^1 (v'^2(x) + v^2(x))\,dx}\,.$$

Let the bilinear form $a(\cdot,\cdot)$ be defined by

$$a(v,u) = \int_0^1 4\,v'(x)\,u'(x)\,dx \qquad \forall\, v,u \in V.$$

- Show that this bilinear form is symmetric.
- Then show that, for some positive constants α and γ, one has

  $$\alpha\|v\|^2 \le a(v,v) \le \gamma\|v\|^2 \qquad \forall\, v \in V.$$

  Specify the constants α and γ.
- Let $L(v) = \int_0^1 v(x)\,f(x)\,dx$ for any $v \in V$. Consider the following variational problem and its discretization:
  - *Variational problem (VAR)*:
    Find $u \in V$ such that $a(u,v) = L(v) \qquad \forall\, v \in V$.
  - *Discrete Variational problem (VAR^h)*:
    Let V^h be a finite-dimensional subspace of V.
    Find $u_h \in V^h$ such that $a(u_h, v_h) = L(v_h) \qquad \forall\, v_h \in V^h$.

  Prove that for the solution u of problem (VAR) and the solution u_h of problem (VAR^h), we have

  $$a(u-u_h, u-u_h) \le a(u-v_h, u-v_h) \qquad \forall\, v_h \in V^h.$$
- Then prove that

  $$\|u-u_h\| \le C\|u-v_h\| \qquad \forall\, v_h \in V^h$$

  and specify the positive constant C here. (Hint: use α and γ).

. . . .

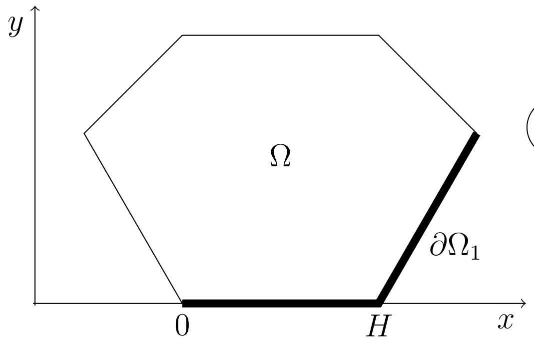
(b) In a two-dimensional domain Ω consider the problem: 15%

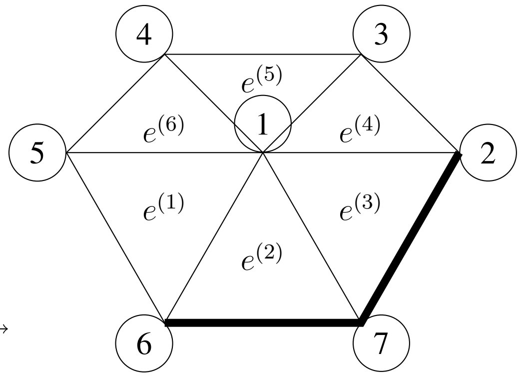
$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f \quad \text{for } (x,y) \in \Omega,$$

$$u(x,y) = 0 \quad \text{for } (x,y) \in \partial\Omega_1, \qquad \frac{\partial u(x,y)}{\partial \mathbf{n}} = 0 \quad \text{for } (x,y) \in \partial\Omega_2,$$

where f is constant and $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ is the boundary of Ω.

This problem is discretized using *linear finite elements*, where the domain Ω and its triangulation are as follows:



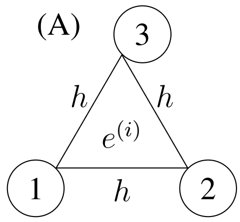


Here $e^{(1)}$, $e^{(2)}$ and $e^{(3)}$ are equilateral triangles, while $e^{(4)}$, $e^{(5)}$ and $e^{(6)}$ are right-angled triangles.

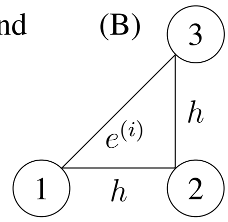
For this discretization:

- Find the global stiffness matrix $K_{(f)}$ and the global load vector $F_{(f)}$ in which the boundary conditions are ignored.
- Find the global stiffness matrix K and the global load vector F which take the boundary conditions into consideration.
- Then write the numerical method as a linear system $KU = F$. For each entry of the unknown vector U specify with which mesh node it is associated.

Note that for the linear elements (A)



and (B)



the local stiffness matrix $K^{(i)}$ and the local load vector $F^{(i)}$ are respectively given by

$$\text{(A)}\qquad K^{(i)} = \frac{1}{2\sqrt{3}}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \qquad F^{(i)} = \frac{fh^2\sqrt{3}}{12}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\text{(B)}\qquad K^{(i)} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad F^{(i)} = \frac{fh^2}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

. . . .

(c) Consider the problem: 15%

$$-u'' + 5u' + 7u = f \quad \text{for } x \in (0,1), \qquad u(0) = u'(1) = 0.$$

- Obtain a weak formulation of this problem.
(Note: you are expected to specify the space in which u is found, from which space arbitrary functions v are taken, and which boundary conditions u and v are required to satisfy if any.)
- Suppose this problem is discretized using piecewise **linear** finite elements with the local shape functions $\phi_1^{(i)}$ and $\phi_2^{(i)}$ defined on each element $e^{(i)} = (x_i, x_{i+1})$ by

$$\phi_k^{(i)} = \varphi_k\left(\frac{x - x_i}{h_i}\right), \quad k = 1, 2,$$

where $h_i = x_{i+1} - x_i$ and

$$\varphi_1(t) = 1 - t, \qquad \varphi_2(t) = t\,.$$

Find the local stiffness matrix $K^{(i)}$ and the local load vector $F^{(i)}$, assuming that $f = \text{const}$.

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2 Answer parts (a), (d) and any *two* of parts (b), (c), (e). **25%**

Let ε be a positive constant. In the square domain $\Omega = (0,1) \times (0,1)$ with the boundary $\partial\Omega$ consider the problem:

$$Lu = -\varepsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - 7\,\frac{\partial u}{\partial y} = f(x,y) \qquad \text{for } (x,y) \in \Omega,$$

$$u(x,y) = 0 \qquad \text{for } (x,y) \in \partial\Omega.$$

This problem is discretized on the uniform mesh $\{(x_i, y_j)\}_{i,j=1,\ldots,N+1}$, where $x_i = (i-1)h,\ y_j = (j-1)h,\ h = 1/N$, by the finite difference method:

$$L^h U_{ij} = -\varepsilon\,\frac{U_{i-1,j} + U_{i+1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{ij}}{h^2} - 7\,\frac{U_{i,j+1} - U_{i,j-1}}{2h} = f(x_i, y_j)$$

for $i, j = 2, \ldots, N$, with the boundary conditions:

$$U_{ij} = 0 \quad \text{for } (x_i, y_j) \in \partial\Omega.$$

. . . .

(a) Estimate the local truncation error r_{ij} of this method associated with the mesh node (x_i, y_j). 5%

. . . .

(b) Show that the finite difference operator L^h, possibly under a certain condition that involves h and ε, satisfies the discrete maximum principle in the form: 6%

$$\left.\begin{array}{rl} L^h V_{ij} \le 0 & \forall\, i,j = 2,\dots,N \\ V_{ij} \le 0 & \forall\, (x_i,y_j) \in \partial\Omega \end{array}\right\} \Rightarrow\ V_{ij} \le 0\ \forall\, i,j = 1,\dots,N+1.$$

Specify the discrete-maximum-principle condition on h, if there is any.

. . . .

(c) Using the result of part (b), show that the finite difference operator L^h satisfies the discrete comparison principle in the form: 6%

$$\left.\begin{array}{rl} |L^h W_{ij}| \le L^h V_{ij} & \forall\, i,j = 2,\dots,N \\ |W_{ij}| \le V_{ij} & \forall\, (x_i,y_j) \in \partial\Omega \end{array}\right\} \Rightarrow\ |W_{ij}| \le V_{ij}\ \forall\, i,j = 1,\dots,N+1.$$

. . . .

(d) Using the discrete comparison principle described in part (c), show that 8%

$$W_{ij} = 0\ \ \forall\, (x_i,y_j) \in \partial\Omega\ \Rightarrow\ \max_{i,j=1,\dots,N+1} \bigl|W_{ij}\bigr| \le C_0 \max_{i,j=2,\dots,N} \bigl|L^h W_{ij}\bigr|$$

for some positive constant C_0. Specify this constant.

. . . .

(e) Using the result of part (a) and the property described in part (d), estimate the error of the finite difference method 6%

$$\max_{i,j=1,\dots,N+1} \bigl|U_{ij} - u(x_i,y_j)\bigr|,$$

where U_{ij} is the computed solution, and $u(x_i,y_j)$ is the exact solution at the mesh node (x_i, y_j).

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3 Answer parts (a), (d) and *one* of parts (b) and (c). **20%**

Consider the problem

$$\begin{aligned} u_t &= u_{xx} && \text{for } x \in (0,1),\ t > 0, \\ u(0,t) = u(1,t) &= 0 && \text{for } t > 0, \\ u(x,0) &= g(x) && \text{for } x \in [0,1]. \end{aligned}$$

This problem is discretized on the uniform mesh

$$\{(x_j, t_m),\ j = 1, ..., N+1,\ m = 1, 2, \dots\},$$

where $x_j = (j-1)h$ with $h = 1/N$, and $t_m = (m-1)k$. Let U_j^m be the computed solution associated with the point (x_j, t_m).

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(a) Using Von Neumann's method, prove that the Du Fort-Frankel method 9%

$$\frac{U_j^{m+1} - U_j^{m-1}}{2k} = \frac{U_{j-1}^m - U_j^{m+1} - U_j^{m-1} + U_{j+1}^m}{h^2},$$

is *unconditionally stable*.

. . . .

(b) Using Von Neumann's method, find out whether the following method is unconditionally stable, unconditionally unstable, or conditionally stable. If it is conditionally stable, find the stability condition. 9%

$$\frac{U_j^{m+1} - U_j^m}{k} = 0.25\,\frac{U_{j-1}^{m+1} - 2U_j^{m+1} + U_{j+1}^{m+1}}{h^2} + 0.75\,\frac{U_{j-1}^m - 2U_j^m + U_{j+1}^m}{h^2}.$$

. . . .

(c) Estimate the local truncation errors of the methods in parts (a) and (b). 9%

. . . .

(d) For each of the methods in parts (a) and (b), specify whether it is implicit or explicit. Explain your answer. 2%
