List of Questions to prepare for the Final Exam: Finite Element Method

Consider the problems:

(a) $-5u'' + 2u = f, \quad x \in (0,1), \quad u'(0) = u'(1) = 0;$

(b) $-3u'' + 2u = f, \quad x \in (0,1), \quad u(0) = u(1) = 0;$

(c) $-3u'' = f, \quad x \in (0,1), \quad u(0) = u(1) = 0;$

(e) $-u'' + 7u = f, \quad x \in (0,1), \quad u(0) = u'(1) = 0;$

(f) $-u'' = f, \quad x \in (0,1), \quad u'(0) = u(1) = 0;$

(g) $-\Delta u + u = f, \quad x \in \Omega \subset R^2, \quad u(x)\Big|_{x\in\partial\Omega} = 0;$

(h) $-3\Delta u + u = f, \quad x \in \Omega \subset R^2, \quad u(x)\Big|_{x\in\partial\Omega} = 0;$

(i) $-2\Delta u + 3u = f, \quad x \in \Omega \subset R^2, \quad u(x)\Big|_{x\in\partial\Omega} = 0;$

(I) $-u'' + 6u' = f, \quad x \in (0,1), \quad u(0) = u'(1) = 0;$

(II) $-u'' + 3u' = f, \quad x \in (0,1), \quad u(0) = u(1) = 0;$

(III) $-2u'' + 5u' + u = f, \quad x \in (0,1), \quad u(0) = u'(1) = 0.$

....

1. For each of the problems (a)-(i) and (I)-(III) obtain its weak formulation. (Hint: multiply the differential equation by an arbitrary function v and integrate using integration by parts and the boundary conditions. Note: you are expected to specify the space in which u is found and from which arbitrary functions v are taken an the boundary conditions that they satisfy, if any.)

....

2. For each of problems (a)-(i) give their weak formulations in the form:

$$\text{Find } u \in V: \qquad a(u,v) = L(v) \qquad \forall\, v \in V,$$

specifying V, $a(\cdot,\cdot)$, and $L(\cdot)$.

Furthermore, check that $a(\cdot,\cdot)$ and $L(\cdot)$ satisfy assumptions (A1)-(A3)—see p. 1 of §4.5—and specify the constants γ and α in these assumptions.

....

. . . .

3. Using piecewise linear elements (see §4.2) with the local shape functions

$$\phi_i^{(i)} = \frac{(x_{i+1} - x)}{h_i}, \quad \phi_{i+1}^{(i)} = \frac{(x - x_i)}{h_i} \qquad \text{for} \quad x \in e^{(i)} = [x_i, x_{i+1}].$$

find the local stiffness matrix $K^{(i)}$ and the local load vector $F^{(i)}$ for equations in problems (a)-(e) and (I)-(III), where $f = \text{const}$.

. . . .

4. Using piecewise quadratic finite elements with the local shape functions $\phi_1^{(i)}$, $\phi_2^{(i)}$, $\phi_3^{(i)}$ defined in §4.3, find the local stiffness matrix $K^{(i)}$ and the local load vector $F^{(i)}$ for equations in problems (c), (I)-(III), where $f = \text{const}$.

. . . .

5. For problems (c), (f), (I), (II), where $f = \text{const}$,

   on each of the meshes with
   (i) $h_1 = h_2 = 1/4$, $h_3 = 1/2$; (ii) $h_1 = 1/4$, $h_2 = 1/2$, $h_3 = 1/4$,

   consider the finite element methods using:

   - linear elements; see §4.2;
   - quadratic elements; see §4.3;
   - linear elements on $e^{(1)}$ and $e^{(2)}$ and quadratic elements on $e^{(3)}$.

   (i) Find $K^{(f)}$ and $F^{(f)}$.
   (ii) Find K and F and then write the numerical method as a linear system $KU = F$.
   (iii) For each entry of the unknown vector U specify to which mesh node it is assigned.

   Hint: use local matrices $K^{(i)}$ and $F^{(i)}$ obtained in the previous exercises.

. . . .

In Ω consider the Poisson equation:

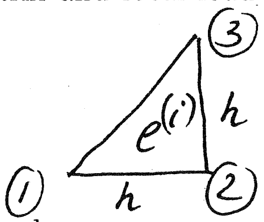
$$-\Delta u \equiv -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f, \quad (x,y) \in \Omega, \tag{1}$$

where $f = \text{const}$, with the boundary conditions

$$u(x,y) = 0, \quad (x,y) \in \partial\Omega_1; \qquad \frac{\partial u(x,y)}{\partial \mathbf{n}} = 0, \quad (x,y) \in \partial\Omega_2; \tag{2}$$

where $\mathbf{n}$ is the outward normal to $\partial\Omega_2$.

6. The local stiffness matrix and local load vector of the linear element

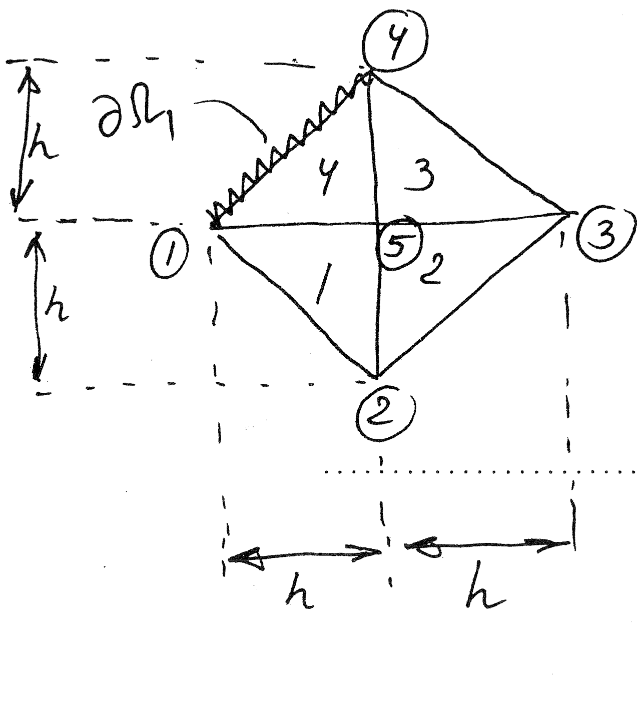


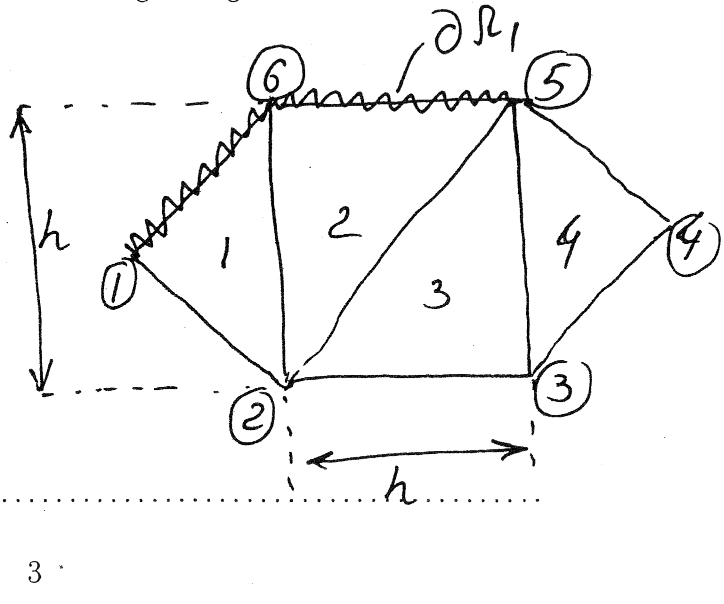
(see § 4.4)

for equation (1) is given by

$$K^{(i)} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad F^{(i)} = \frac{fh^2}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

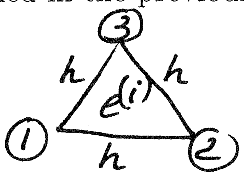
Construct the global stiffness matrix $K_{(f)}$ and the global load vector $F_{(f)}$, and then K and F for the following triangulations:



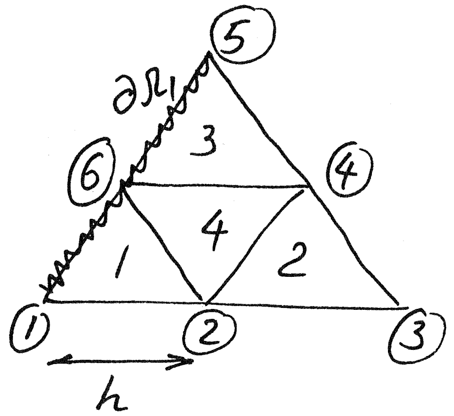


7. Do exercises on p.19 of §4.4.

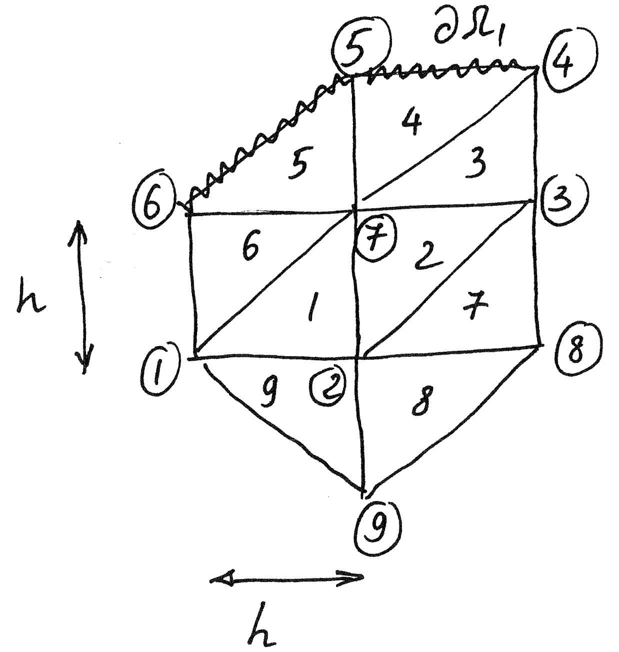
8. Using $K^{(i)}$ and $F^{(i)}$ obtained in the previous exercise for the element:



find $K_{(f)}$, $F_{(f)}$, and then K and F for the triangulation



9. Suppose you are given $K_{(f)}$ and $F_{(f)}$ for triangulations from Example 1 (§4.4 , p.13). Modifying them, obtain the matrices $K_{(f)}$ and $F_{(f)}$, and then K and F for the triangulations:



10. Prove each of the following statements (problems VAR, MIN, VAR^h, and MIN^h are defined on p. 1-3 of §4.5):

   (a) If u is a solution of problem VAR, then u is a solution of problem MIN.

   (b) If u is a solution of problem MIN, then u is a solution of problem VAR.

   (c) If u_h is a solution of problem VAR^h, then u_h is a solution of problem MIN^h.

   (d) If u_h is a solution of problem MIN^h, then u_h is a solution of problem VAR^h.

   (e) For the solution u of problem VAR and the solution u_h of problem VAR^h, we have

   $$\|u - u_h\|_a \leq \|u - v_h\|_a \qquad \forall v_h \in V^h,$$

   where $\|v\|_a = \sqrt{a(v, v)}$ is the energy norm.

   (f) For the solution u of problem VAR and the solution u_h of problem VAR^h, we have

   $$\|u - u_h\| \leq C\|u - v_h\| \qquad \forall v_h \in V^h,$$

   where $\|v\|$ is the norm of the space V, in which problem VAR is stated.
   (Hint: first prove the previous statement, then use assumptions (A1)-(A3)—see §4.5, p. 1—on $a(\cdot, \cdot)$.)
   Furthermore, specify the constant C here in terms of γ and α from (A1)-(A3).