UNIVERSITY OF LIMERICK
OLLSCOIL LUIMNIGH

Faculty of Science and Engineering

Department of Mathematics & Statistics

END OF SEMESTER ASSESSMENT PAPER

MODULE TITLE: Numerical Partial Differential Equations

MODULE CODE: MS4008

SEMESTER: Autumn 2021

LECTURER/EXAMINER: Prof. N. Kopteva

EXTERNAL EXAMINER: Prof. R. E. Wilson

DURATION OF EXAMINATION: $2\frac{1}{2}$ hours

PERCENTAGE OF TOTAL MARKS: 75% (+25 % for continuous assessment)

INSTRUCTIONS TO CANDIDATES:

- **Answer questions 1, 2, and 3. To obtain maximum marks you must show all your work clearly and in detail.**
- **Standard mathematical tables are provided by the invigilators. Under no circumstances should you use your own tables or be in possession of any written material other than that provided by the invigilators.**
- **Non-programmable, non-graphical calculators that have been approved by the lecturer are permitted.**

1. Answer parts (a) and (b), and one of parts (c) and (d). **30%**

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(a) The norm in the space $H^1(0,1)$ is given by 15%

$$\|v\| = \sqrt{\int_0^1 (v'^2(x) + v^2(x))\,dx}\ .$$

Consider the problem:

$$-7u'' + 3x^2 u = f \quad \text{for} \quad x \in (0,1), \qquad u'(0) = u(1) = 0.$$

i. Obtain a weak formulation of this problem in the form of

- *Variational problem (VAR)*:
  Find $u \in V$ such that $a(u,v) = L(v) \quad \forall v \in V$.

Here $a(\cdot,\cdot)$ is a bilinear form, $L(\cdot)$ is a linear form, V is a subspace of $H^1(0,1)$; and their form should be specified.

ii. Show that the bilinear form $a(\cdot,\cdot)$ is symmetric.

iii. Then show that, for some positive constants α and γ, one has

$$\alpha\|v\|^2 \le a(v,v) \le \gamma\|v\|^2 \quad \text{for all} \quad v \in V.$$

Specify the constants α and γ.

iv. Using the properties of $a(\cdot,\cdot)$ that you have established, prove that the above *Variational problem (VAR)* is equivalent to the following

- *Minimization problem (MIN)*:
  Find $u \in V$ such that $F(u) \le F(v) \quad \forall v \in V$,
  where $F(v) = \frac{1}{2}a(v,v) - L(v)$.

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(b) Obtain a weak formulation of the following problem: 3%

$$-u'' - 2u' = f \quad \text{for} \quad x \in (0,1), \qquad u(0) = 3, \quad u'(1) = 5u(1) + 7.$$

Note: you are expected to specify the space in which u is found, from which space the arbitrary functions v are taken, and which boundary conditions u and v are required to satisfy, if any.

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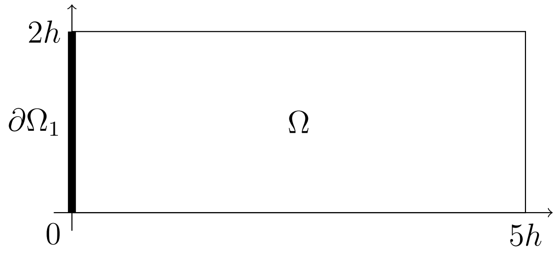
(c) Consider the problem: 12%

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f \quad \text{for } (x,y) \in \Omega,$$

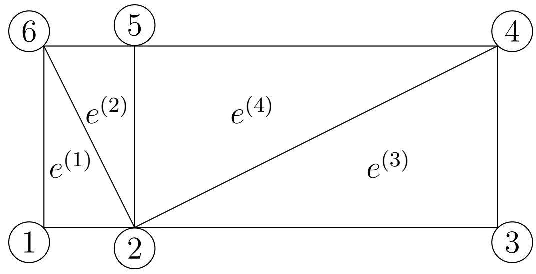
$$u(x,y) = 0 \quad \text{for } (x,y) \in \partial\Omega_1, \qquad \frac{\partial u(x,y)}{\partial \mathbf{n}} = 0 \quad \text{for } (x,y) \in \partial\Omega_2,$$

where f is constant, Ω is a two-dimensional domain, the boundary of which $\partial\Omega$ is the union of disjoint sets $\partial\Omega_1$ and $\partial\Omega_2$.

This problem is posed in the domain $\Omega = (0, 5h) \times (0, 2h)$, while $\partial\Omega_1$ is the part of the boundary that overlaps with the line $x = 0$:



The above problem is discretized using *linear finite elements* with the following triangulation:



Each $e^{(i)}$, for $i = 1, \dots, 4$, is a right-angled triangle.

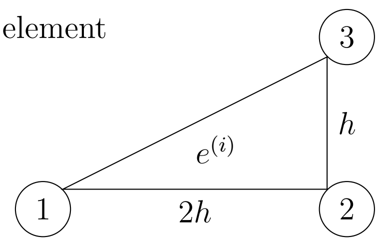
For this discretization:

i. Find the global stiffness matrix $K_{(f)}$ and the global load vector $F_{(f)}$ in which the boundary conditions are still to be addressed.

ii. Find the global stiffness matrix K and the global load vector F which take the boundary conditions into consideration.

iii. Then write the numerical method as a linear system $KU = F$. For each entry of the unknown vector U, specify with which mesh node it is associated.

Note that for the linear element



the local stiffness matrix $K^{(i)}$ and the local load vector $F^{(i)}$ are given by

$$K^{(i)} = \frac{1}{4}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 5 & -4 \\ 0 & -4 & 4 \end{bmatrix}, \qquad F^{(i)} = \frac{f}{3}\begin{bmatrix} h^2 \\ h^2 \\ h^2 \end{bmatrix}.$$

....

(d) Recall the problem from part (b): 12%

$$-u'' - 2u' = f \quad \text{for} \ \ x \in (0,1), \qquad u(0) = 3, \quad u'(1) = 5u(1) + 7.$$

Suppose this problem is discretized using piecewise quadratic finite elements with the local shape functions $\phi_1^{(i)}$, $\phi_2^{(i)}$, $\phi_3^{(i)}$ defined on each mesh element $e^{(i)} = (x_i, x_{i+1})$ by

$$\phi_k^{(i)}(x) = \varphi_k\left(\frac{x - x_i}{h_i}\right), \quad k = 1, 2, 3,$$

where $h_i = x_{i+1} - x_i$ and

$$\varphi_1(t) = (t-1)(2t-1), \quad \varphi_2(t) = 4t(1-t), \quad \varphi_3(t) = t(2t-1).$$

Find the local stiffness matrix $K^{(i)}$. and the local load vector $F^{(i)}$, assuming that f is constant.

NOTE: You may use the fact that the local stiffness matrix for the equation $-u'' = f$ is $\frac{1}{h_i}\begin{bmatrix} \frac{7}{3} & -\frac{8}{3} & \frac{1}{3} \\ -\frac{8}{3} & \frac{16}{3} & -\frac{8}{3} \\ \frac{1}{3} & -\frac{8}{3} & \frac{7}{3} \end{bmatrix}$

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2. Answer parts (a), (b), (d), and one of parts (c) and (e). **25%**

Let a and b be positive constants. In the square domain $\Omega = (0,1) \times (0,1)$ with the boundary $\partial\Omega$ consider the problem:

$$Lu = -a\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + b\,\frac{\partial u}{\partial y} = f(x,y) \qquad \text{for } (x,y) \in \Omega,$$

$$u(x,y) = 0 \qquad \text{for } (x,y) \in \partial\Omega.$$

This problem is discretized on the uniform mesh $\{(x_i, y_j)\}_{i,j=1,\ldots,N+1}$, where $x_i = (i-1)h$, $y_j = (j-1)h$, $h = 1/N$, by the finite difference method:

$$L^h U_{i,j} = -a\left(\frac{U_{i-1,j} + U_{i+1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{ij}}{h^2}\right) + b\left(\frac{U_{i,j+1} - U_{i,j-1}}{2h}\right) = f(x_i, y_j)$$

for $i, j = 2, \ldots, N$, with the boundary conditions:

$$U_{ij} = 0 \quad \text{for } (x_i, y_j) \in \partial\Omega.$$

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(a) Specify the local truncation error r_{ij} of this method associated with the mesh node (x_i, y_j). Obtain a truncation error estimate of the form $r_{ij} = O(h^p)$ for some $p > 0$. 5%

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(b) Show that the finite difference operator L^h, possibly under a certain condition that involves h, a, and b, satisfies the discrete maximum principle of the form: 6%

$$\left.\begin{array}{r} L^h V_{ij} \le 0 \quad \forall\, i,j = 2,\dots,N \\ V_{ij} \le 0 \quad \forall\, (x_i, y_j) \in \partial\Omega \end{array}\right\} \Rightarrow \; V_{ij} \le 0 \;\forall\, i,j = 1,\dots,N+1.$$

Specify the discrete maximum principle condition on h, if one was used in your proof.

. . . .

(c) Using the result of part (b), show that the finite difference operator L^h satisfies the discrete comparison principle of the form: 6%

$$\left.\begin{array}{ll} |L^h W_{ij}| \le L^h V_{ij} & \forall\, i,j = 2,\dots,N \\ |W_{ij}| \le V_{ij} & \forall\, (x_i, y_j) \in \partial\Omega \end{array}\right\} \Rightarrow \; |W_{ij}| \le V_{ij} \;\forall\, i,j = 1,\dots,N+1.$$

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(d) Using the discrete comparison principle described in part (c), show that 8%

$$W_{ij} = 0 \;\;\forall\, (x_i, y_j) \in \partial\Omega \;\Rightarrow\; \max_{i,j=1,\dots,N+1} \bigl|W_{ij}\bigr| \le C_0 \max_{i,j=2,\dots,N} \bigl|L^h W_{ij}\bigr|$$

for some positive constant C_0. Specify this constant.

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(e) Using the result of part (a) and the property described in part (d), estimate the error of the finite difference method 6%

$$\max_{i,j=1,\dots,N+1} \bigl|U_{ij} - u(x_i, y_j)\bigr|,$$

where U_{ij} is the computed solution and $u(x_i, y_j)$ is the exact solution at the mesh node (x_i, y_j).

3. Answer parts (a) and (d), and one of parts (b) and (c). **20%**

Consider the problem

$$\begin{aligned} u_t &= u_{xx} && \text{for } x \in (0,1),\ t > 0, \\ u(0,t) = u(1,t) &= 0 && \text{for } t > 0, \\ u(x,0) &= g(x) && \text{for } x \in [0,1]. \end{aligned}$$

This problem is discretized on the uniform mesh

$$\{(x_j, t_m),\ j = 1, ..., N+1,\ m = 1, 2, \ldots\},$$

where $x_j = (j-1)h$ with $h = 1/N$, and $t_m = (m-1)k$. Let U_j^m be the computed solution associated with the point (x_j, t_m).

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(a) Using Von Neumann's method, prove that the Du Fort-Frankel method 9%

$$\frac{U_j^{m+1} - U_j^{m-1}}{2k} = \frac{U_{j-1}^m - U_j^{m+1} - U_j^{m-1} + U_{j+1}^m}{h^2},$$

is *unconditionally stable.*

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(b) Using Von Neumann's method, find out whether the following method is unconditionally stable, unconditionally unstable, or conditionally stable. If it is conditionally stable, find the stability condition. 9%

$$\frac{U_j^{m+1} - U_j^m}{k} = 0.2\,\frac{U_{j-1}^{m+1} - 2U_j^{m+1} + U_{j+1}^{m+1}}{h^2} + 0.8\,\frac{U_{j-1}^m - 2U_j^m + U_{j+1}^m}{h^2}.$$

. . . .

(c) Estimate the local truncation errors of the methods in parts (a) and (b). 9%

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(d) For each of the methods in parts (a) and (b), specify whether it is implicit or explicit. Explain your answer. 2%
