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A STUDY OF DIFFERENCE SCHEMES WITH THE FIRST DERIVATIVE APPROXIMATED BY A CENTRAL DIFFERENCE RATIO†

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For an ordinary second-order differential equation in which the coefficient of the highest derivative is a small parameter, the classical difference scheme which uses a central difference ratio to approximate the first derivative is investigated. By means of a detailed analysis of Green's function of the grid problem, it is established that the scheme is solvable on Shishkin's piecewise-uniform grid which clusters in the boundary layer and has uniform accuracy $O(N^{-2}\ln^2 N)$ with respect to the small parameter, where N is the number of grid nodes. © 1997 Elsevier Science Ltd. All rights reserved.

INTRODUCTION

For a singularly perturbed ordinary second-order differential equation, we will consider the simplest two-point boundary-value problem‡

$$Lu \equiv -\varepsilon(p(x)u')' - r(x)u' = f(x), \quad 0 < x < 1, \tag{1}$$

$$u(0) = g_0, \quad u(1) = g_1,$$

where

$$p(x) \geqslant p_0 = \text{const} > 0, \quad r(x) \geqslant r_0 = \text{const} > 0, \tag{2}$$

and $\varepsilon \in (0, 1]$ is a small parameter. It is well known [1] that as $\varepsilon \to 0$ in the half-interval $0 < x \leqslant 1$ the solution of this problem converges to the solution of the degenerate problem

$$-r(x)\ v' = f(x), \quad v(1) = g_1,$$

and for small ε the boundary condition, unused in the degenerate problem, leads to the formation in the neighbourhood of the point $x = 0$ of a so-called boundary layer, where the solution $u(x)$ of the original problem (1) varies strongly.

The presence of the small parameter and the boundary layer that it generates leads to considerable difficulties in the numerical solution of problem (1) and others like it (see [2, 3], for example). Attempts have been made to overcome these in at least two ways: by developing "adjusting schemes" [2, 4] which converge uniformly with respect to the small parameter on arbitrary grids, and by using special non-uniform grids which cluster in the boundary layer, for ordinary difference schemes [5, 6].

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‡ After the paper had gone to press, we established that the results also apply to the equation $Lu+qu=f(x)$ with $q(x)\geqslant 0$, which is more general than (1).

Let $\bar{\omega}=\{x_i\,|\,0=x_0<\ldots<x_N=1\}$ be an arbitrary non-uniform grid in [0, 1]. As usual we put

$$h_i = x_i - x_{i-1},\quad \hbar_i = (h_i + h_{i+1})/2,\quad v_{\bar{x}i} = (v_i - v_{i-1})/h_i,$$
$$v_{xi} = v_{\bar{x},i+1},\quad v_{\hat{x}i} = (v_i - v_{i-1})/\hbar_i,\quad v_{\hat{x}i} = (v_{i+1} - v_i)/\hbar_i$$

and on the grid $\bar{\omega}$ construct the classical approximation of problem (1)

$$-\varepsilon(p^h u^h_{\bar{x}})_{\hat{x}i} - r^h_i(\sigma u^h_{\bar{x}} + (1-\sigma)u^h_x)_i = f^h_i,\quad i = 1,2,\ldots,N-1, \tag{3}$$
$$u^h_0 = g_0,\quad u^h_N = g_1,$$

where $\sigma=\text{const}$ is a parameter of the scheme, and

$$p^h_i = p(x_i - h_i/2),\quad r^h_i = r(x_i),\quad f^h_i = f(x_i). \tag{4}$$

We know (see [7] for example) that if the grid $\bar{\omega}$ is uniform, that is, all $h_i=h=1/N$, on smooth solutions of Eq. (1) the approximation error of difference scheme (3) with $\sigma=1/2$ is $O(h^2)$, whereas it is only $O(h)$ when $\sigma=1$. However, on an arbitrary grid with small ε for $\sigma=1$ scheme (3) is monotone and thus conforms to the maximum principle [7], but for $\sigma=1/2$, the maximum principle only applies to (3) provided that the parameter ε is not too small compared with h_i. The absence of a maximum principle in scheme (3) for $\sigma=1/2$ leads to a "sawtoothed" solution, which has given it a bad reputation amongst computer users. It should be noted, however, that if the grid $\bar{\omega}$ has not been specially adapted to solve problem (1) for small ε, scheme (3) is not uniformly convergent with respect to ε for either $\sigma=1/2$ or $\sigma=1$. Even so, the fact, established in [8], that outside the boundary layer on a uniform grid $\bar{\omega}$, scheme (3) with $\sigma=1$ converges at a rate $O(h)$, is an argument in its favour. This is not the case when $\sigma=1/2$, as simple examples show.

The uniform convergence of scheme (3) with respect to ε on the entire grid was first investigated by Shishkin (see [6], for example), who introduced a piecewise-uniform grid

$$\Omega = \{x_i | x_i = ih,\ i=1,2,\ldots,n;\ x_i = x_n + (i-n)H,\ i=n+1,\ldots,N-1;\ h=\delta/n, \tag{5}$$
$$H = (1-\delta)/(N-n),\ \delta = \min(C\varepsilon\ln N, A),\ N/n = B = O(1),\ 0<A<1\},$$

which clusters in the boundary layer, and proved that on grid (5) for $\sigma=1$ with $C>p(0)/r(0)$ scheme (3) converges uniformly with respect to ε in the sense of the grid norm $C(\Omega)$ at a rate $O(N^{-1}\ln^2 N)$. A modification of the monotone scheme of Samarskii [7] was constructed in [3] and its uniform convergence with respect to ε at a rate $O(N^{-2}\ln^2 N)$ in the same norm was proved on grid (5) with $C>2p(0)/r(0)$.

Below we shall investigate scheme (3) with $\sigma=1/2$ on the same grid (5). We will rewrite it in the form

$$(L^h u^h)_i \equiv -\varepsilon(p^h u^h_{\bar{x}})_{\hat{x}i} - r^h_i u^h_{\overset{\circ}{x}i} = f^h_i,\quad i=1,2,\ldots,N-1, \tag{6}$$
$$u^h_0 = g_0,\quad u^h_N = g_1,$$

where $v_{\overset{\circ}{x}i}=(v_{i+1}-v_{i-1})/(2\hbar_i)$ is the central difference ratio.

Theorem 1. Let $u(x)$ be a solution of problem (1), (2) with sufficiently smooth coefficients and right-hand side, and let u^h be a solution of (6), (4) on the Shishkin grid (5). Then if the parameter C of the grid Ω satisfies the condition

$$C > 2p(0)/r(0), \tag{7}$$

and $N \geqslant N_0(p(x), r(x))$, then

$$\max_i |u(x_i) - u^h(x_i)| = O(N^{-2}\ln^2 N)$$

uniformly with respect to ε.

The present paper is devoted to the proof of this theorem.

1. GREEN'S GRID FUNCTION

The key to the proof of Theorem 1 is the proof of uniform boundedness with respect to ε of Green's function $G(x_i, \xi_j)$ of problem (6). As a function of x_i for fixed ξ_j, Green's function is defined by the relations

$$L^h G(x_i,\xi_j)=\delta^h(x_i,\xi_j),\quad x_i\in\Omega,\quad \xi_j\in\Omega, \tag{1.1}$$

$$G(0,\xi_j)=G(1,\xi_j)=0,\quad \xi_j\in\Omega, \tag{1.2}$$

where

$$\delta^h(x_i,\xi_j)=\begin{cases}\hbar_i^{-1} & \text{for } x_i=\xi_j,\\ 0 & \text{for } x_i\neq\xi_j\end{cases}$$

is the grid analogue of Dirac's delta function.

Let

$$(u,\upsilon)=\sum_{j=1}^{N-1} u_j \upsilon_j \hbar_j \tag{1.3}$$

be the scalar product defined for functions $u(x_j)$ and $v(x_j)$ which are assigned on $\bar{\omega}$ and vanish for $j=0$ and $j=N$. Then if $v^h(x_i)$ is a solution of problem (6) with homogeneous boundary conditions $v^h(0)=v^h(1)=0$, we have

$$\upsilon^h(x_i)=(G(x_i,\xi_j),f^h(\xi_j)). \tag{1.4}$$

Let L^{h^*} denote the conjugate grid operator to L^h of (6) in the sense of the scalar product (1.3), that is,

$$(L^h u,\upsilon)=(u,L^{h*}\upsilon).$$

It is easily verified that

$$(L^{h*}\upsilon)_j\equiv-\varepsilon(p^h\upsilon_{\bar{\xi}})_{\hat{\xi}j}+(r^h\upsilon)_{\overset{\circ}{\xi}j}. \tag{1.5}$$

The operator L^{h^*} can be used to describe the function $G(x_i,\xi_j)$ as a function of ξ_j for fixed x_i. Thus:

$$L^{h*}G(x_i,\xi_j)=\delta^h(\xi_j,x_i),\quad \xi_j\in\Omega,\ x_i\in\Omega, \tag{1.6}$$

$$G(x_i,0)=G(x_i,1)=0,\quad x_i\in\Omega.$$

We will construct the function $G(x_i,\xi_j)$ in explicit form. To do so, we consider the function $\alpha(x_i)=\alpha_i$, which is a solution of the following Cauchy grid problem:

$$L^h\alpha_i=0,\quad x_i\in\Omega,\quad \alpha_0=0,\quad \left(\varepsilon p_1^h+\frac{h_1}{2}r_0^h\right)\alpha_{x,0}=1. \tag{1.7}$$

Putting here

$$\varepsilon(p^h\alpha_{\bar{x}})_i=w_i,\quad i=1,2,\ldots,N, \tag{1.8}$$

and bearing (6) in mind, we find that w_i satisfies the recurrence relation

$$w_{i+1}-w_i+\frac{r_i^h}{2}\left(\frac{h_{i+1}}{\varepsilon p_{i+1}^h}w_{i+1}+\frac{h_i}{\varepsilon p_i^h}w_i\right)=0. \tag{1.9}$$

Hence

$$w_{i+1}=q_i w_i, \tag{1.10}$$

where

$$q_i=\left(1-\frac{r_i^h h_i}{2\varepsilon p_i^h}\right)\left(1+\frac{r_i^h h_{i+1}}{2\varepsilon p_{i+1}^h}\right)^{-1},\quad i=1,2,\ldots,N-1, \tag{1.11}$$

and therefore

$$w_i=w_1\prod_{k=1}^{i-1}q_k,\quad i=1,2,\ldots,N.$$

We now define w_i and q_i for $i=0$, putting $i=0$ and $h_0=0$ in (1.9) and (1.11). Then, remembering the second initial condition (1.7), we will have

$$w_i = \prod_{k=0}^{i-1} q_k, \quad i=0,1,\dots,N. \tag{1.12}$$

From (1.8) and the first initial condition of (1.7) we find

$$\alpha_i = \frac{1}{\varepsilon}\sum_{l=1}^{i} \frac{w_l}{p_l^h} h_l, \quad i=0,1,\dots,N. \tag{1.13}$$

We now construct $G(x_i, \xi_j)$. Since in addition to α_i the function $\bar{\alpha}_i = \text{const}$ is also a solution of Eq. (1.7), Green's function can be sought in the form

$$G(x_i,\xi_j) = c \times \begin{cases} \mathcal{B}_j \alpha_i, & i \leqslant j, \\ (\alpha_N - \alpha_i)\beta_j, & i \geqslant j. \end{cases}$$

This function satisfies Eq. (1.1) with $x_i \neq \xi_j$ and boundary condition (1.2). From the requirement of a unique representation for $i=j$ we have $\mathcal{B}_j\alpha_j = (\alpha_N - \alpha_j)\beta_j$ and, therefore,

$$G(x_i,\xi_j) = c \times \begin{cases} \alpha_i(\alpha_N - \alpha_j)\beta_j / \alpha_j; & i \leqslant j, \\ (\alpha_N - \alpha_i)\beta_j, & i \geqslant j. \end{cases} \tag{1.14}$$

Now the function $G(x_i, \xi_j)$ satisfies Eq. (1.6) with respect to the variable ξ_j and, therefore, β_j is such that, together with $\bar{\beta}_j = \beta_j/\alpha_j$, it satisfies the equation

$$L^{h*} v_j = 0, \quad \xi_j \in \Omega. \tag{1.15}$$

Since $G(x_i, 0)$ should also vanish and $\beta_0 = \bar{\beta}_0\alpha_0 = 0$, let

$$L^{h*}\beta_j = 0, \quad \xi_j \in \Omega, \quad \beta_0 = 0, \quad \varepsilon p_1^h \beta_{\xi,0} - \frac{h_1}{2}(r^h\beta)_{\xi,0} = 1. \tag{1.16}$$

We will find β_j. It follows from (1.15) and (1.5) that

$$\varepsilon p_{j+1}^h v_{\bar{\xi},j+1} - (r_j^h v_j + r_{j+1}^h v_{j+1})/2 = \text{const}, \quad j=0,1,\dots,N-1. \tag{1.17}$$

If const $=0$, then

$$\left(1 - \frac{r_{j+1}^h h_{j+1}}{2\varepsilon p_{j+1}^h}\right) v_{j+1} = \left(1 + \frac{r_j^h h_{j+1}}{2\varepsilon p_{j+1}^h}\right) v_j$$

or, allowing for (1.11),

$$\left[v_{j+1}\left(1 - \frac{r_{j+1}^h h_{j+1}}{2\varepsilon p_{j+1}^h}\right)\right]^{-1} = q_j \left[v_j\left(1 - \frac{r_j^h h_j}{2\varepsilon p_j^h}\right)\right]^{-1}.$$

Comparing this with (1.10), we conclude that

$$v_j = c\left[w_j\left(1 - \frac{r_j^h h_j}{2\varepsilon p_j^h}\right)\right]^{-1}, \quad j=0,1,\dots,N, \tag{1.18}$$

where w_j is defined by (1.12). From this and (1.12) it follows that v_j, being a solution of Eq. (1.15), cannot take the role of β_j, the solution of problem (1.16), because the condition $v_0=0$ yields a solution which is identically zero. Thus v_j takes the role of $\bar{\beta}_j$ and, therefore, we can put

$$\beta_j = v_j \alpha_j. \tag{1.19}$$

Now substituting (1.19) into the second initial condition of (1.16) and bearing (1.13) and (1.18) in mind, we find that this condition will be satisfied if we put $c=1$ in (1.18).

It remains to determine the constant c in (1.14). To do so we must substitute (1.14) with the value found for β_j into (1.1) and put $i=j$, obtaining the value $c=\alpha_N^{-1}$. Thus, Green's function $G(x_i, \xi_j)$ is completely defined and can be written in the form

$$G(x_i,\xi_j)=\left[w_j\left(1-\frac{r_j^h h_j}{2\varepsilon p_j^h}\right)\alpha_N\right]^{-1}\times\begin{cases}\alpha_i(\alpha_N-\alpha_j), & i\leqslant j,\\ (\alpha_N-\alpha_i)\alpha_j, & i\geqslant j.\end{cases} \tag{1.20}$$

Remark 1. Of course, relation (1.19) could be obtained without recourse to Green's function, simply by solving Eq. (1.17) with const $\neq 0$. A solution of this equation in the form (1.18) with $c=c_j$ can be found by the method of variation of a constant.

It is worth noting that the representation of Green's function in (1.20) is completely proper provided that $[1-r_j^h h_j/(2\varepsilon p_j^h)]\neq 0$ for any $\xi_j\in\Omega$. Otherwise further elucidation is required, as follows.

For $i\leqslant j$ consider the functions

$$W_{ij}=\frac{w_j}{w_i}=\prod_{k=1}^{j-1} q_k \tag{1.21}$$

and

$$A_{ij}=\frac{\alpha_j-\alpha_{i-1}}{w_i}=\frac{1}{\varepsilon}\sum_{l=i}^{j}\frac{W_{il}}{p_l^h}h_l, \tag{1.22}$$

where w_i and α_i are given by relations (1.12) and (1.13), respectively. Obviously for $i\leqslant m<j$

$$W_{ij}=W_{im}W_{mj}=W_{im}q_m W_{m+1,j}, \tag{1.23}$$

and

$$A_{ij}=A_{im}+W_{i,m+1}A_{m+1,j}. \tag{1.24}$$

We will rewrite $G(x_i,\xi_j)$ of (1.20) in terms of W_{ij} and A_{ij}. Since, by (1.11),

$$w_j\left(1-\frac{r_j^h h_j}{2\varepsilon p_j^h}\right)=w_{j+1}\left(1+\frac{r_j^h h_{j+1}}{2\varepsilon p_{j+1}^h}\right),$$

substituting this relation into the denominator of (1.20) and using (1.22) and (1.21), we will have

$$G(x_i,\xi_j)=\left[\left(1+\frac{r_j^h h_{j+1}}{2\varepsilon p_{j+1}^h}\right)\alpha_N\right]^{-1}\times\begin{cases}\alpha_i A_{j+1,N}, & i\leqslant j,\\ \alpha_j A_{i+1,N}W_{j+1,i+1}, & i\geqslant j.\end{cases} \tag{1.25}$$

This is a proper representation of $G(x_i,\xi_j)$ whether or not any of the q_j vanish. Both representations (1.20) and (1.25) of Green's function will be used below.

2. AUXILIARY BOUNDS

Before coming to the direct estimation of the function $G(x_i,\xi_j)$, we will establish some auxiliary estimates. Since the coefficients $p(x)$ and $r(x)$ of Eq. (1) have been assumed to be continuous on [0, 1], they are bounded there. Let

$$p(x)\leqslant\bar{p},\quad r(x)\leqslant\bar{r}. \tag{2.1}$$

Lemma 1. If the coefficients $p(x)$ and $r(x)$ of Eq. (1) satisfy inequalities (2), (2.1), and the number of nodes of the grid Ω satisfies the condition

$$N>N_1,\ \text{where}\ N_1\geqslant\max\left\{3,\frac{\bar{r}}{2p_0}BC\ln N_1\right\}, \tag{2.2}$$

then for $i=1,2,\ldots,n$ the solution of problem (1.7) increases monotonely and satisfies the inequalities

$$\frac{p_0}{\bar{p}\bar{r}}(1-\mathring{q}^i)\leqslant\alpha_i\leqslant\frac{\bar{p}}{p_0 r_0}(1-\bar{q}^i),\quad i=1,2,\ldots,n, \tag{2.3}$$

where

$$\mathring{q} = \left(1 - \frac{h\bar{r}}{2\varepsilon p_0}\right)\left(1 + \frac{h\bar{r}}{2\varepsilon p_0}\right)^{-1}, \quad \bar{q} = \left(1 - \frac{hr_0}{2\varepsilon \bar{p}}\right)\left(1 + \frac{hr_0}{2\varepsilon \bar{p}}\right)^{-1}. \tag{2.4}$$

Also,

$$0 < \frac{\mathring{q}^{i-1}}{1 + h\bar{r}/2(\varepsilon p_0)} \leqslant w_i \leqslant \frac{\bar{q}^{i-1}}{1 + hr_0/(2\varepsilon \bar{p})}, \quad i = 1, 2, \ldots, n. \tag{2.5}$$

Proof. We will first show that, under the given assumptions,

$$\mathring{q} > 0. \tag{2.6}$$

We find from (2.4) that (2.6) will hold if

$$h < 2\varepsilon p_0 / \bar{r}. \tag{2.7}$$

We first find a bound for h. By definition (5), $h = \delta/n = B\delta/N$. If $C\varepsilon \ln N < A$, by (5), $\delta = C\varepsilon \ln N$ and $h = (BC\varepsilon \ln N)/N$. Since the function $N^{-1} \ln N$ is decreasing for $N \geqslant 3$, from (2.2), we have

$$h < (BC\varepsilon \ln N_1)/N_1 \leqslant 2\varepsilon p_0 / \bar{r}.$$

But if $C\varepsilon \ln N \geqslant A$, we have $\delta = A$,

$$h = AB/N \leqslant (BC\varepsilon \ln N)/N,$$

and again, using (2.2), we arrive at (2.7). This proves that $\mathring{q}$ is positive.

Finding bounds for q_k from (1.11) by using (4), (2) and (2.1) and bearing (2.4) and (2.6) in mind, we will have

$$0 < \mathring{q} \leqslant q_i \leqslant \bar{q} < 1, \quad i = 1, 2, \ldots, n-1. \tag{2.8}$$

The use of these inequalities for (1.12) leads to (2.5). By virtue of (1.8), the monotone increase of $\alpha(x_i)$ is a consequence of the fact, which has already been proved, that w_i is positive. Finally, bounds (2.3) follow from (1.13) and (2.5). This proves the lemma.

Lemma 2. Let the coefficients $p(x)$ and $r(x)$ of Eq. (1) be continuously differentiable functions which satisfy conditions (2), (2.1) and

$$|p'(x)| \leqslant c_1, \quad |r'(x)| \leqslant c_1. \tag{2.9}$$

If, in addition, there is a sufficiently large number of nodes in the grid Ω, that is,

$$N \geqslant N_2 = N_2(p, r) \tag{2.10}$$

(cf. (2.26) and (2.23)), then for $j \geqslant i \geqslant n+1$

$$|W_{ij}| \leqslant \bar{p} / p_0, \tag{2.11}$$

$$|A_{ij}| \leqslant \frac{\bar{p}}{p_0 r_0}\left(1 + \frac{Hr_0}{2\varepsilon \bar{p}}\right)c_2, \tag{2.12}$$

where $1 \leqslant c_2 = c_2(p, r)$ (cf. (2.29)).

Proof. As k runs through values from $i \geqslant n+1$ to $j-1$, the behaviour of the function q_k of (1.11) conforms with one of the following scenarios: (1) q_k remains positive for all k; (2) q_k remains negative for all k; (3) q_k changes sign or vanishes. We will consider each of these separately.

1. Let $q_k > 0$ for $i \leqslant k \leqslant j-1$. Since $i \geqslant n+1$, we have

$$0 < q_k = \left(1 - \frac{Hr_k^h}{2\varepsilon p_k^h}\right)\left(1 + \frac{Hr_k^h}{2\varepsilon p_{k+1}^h}\right)^{-1} \leqslant \left(1 - \frac{Hr_0}{2\varepsilon \bar{p}}\right)\left(1 + \frac{Hr_0}{2\varepsilon \bar{p}}\right)^{-1} \equiv Q < 1 \tag{2.13}$$

and, therefore,

$$0 < W_{ij} \leqslant Q^{j-i} \leqslant 1,$$

which does not conflict with (2.11). Then

$$0 \leqslant A_{ij} < \frac{H}{\varepsilon p_0} \sum_{l=i}^{\infty} Q^{l-i} = \frac{H}{\varepsilon p_0} \frac{1}{1-Q} = \frac{\bar{p}}{p_0 r_0}\left(1 + \frac{H r_0}{2\varepsilon \bar{p}}\right), \tag{2.14}$$

which is the same as (2.12) with $c_2 = 1$.

2. Let $q_k < 0$ for $i \leqslant k \leqslant j-1$. Since for any positive a and b the fraction $(-1+a)/(1+b) > -1$, we have

$$0 > q_k = \frac{p_{k+1}^h}{p_k^h}\left(-1 + \frac{2\varepsilon p_k^h}{H r_k^h}\right)\left(1 + \frac{2\varepsilon p_{k+1}^h}{H r_k^h}\right)^{-1} \geqslant -\frac{p_{k+1}^h}{p_k^h} \tag{2.15}$$

and therefore

$$|W_{ij}| \leqslant p_j^h / p_i^h \leqslant \bar{p} / p_0,$$

which is the same as (2.11). Then, using (1.21) and (1.22) we find that

$$A_{ij} = \frac{H}{\varepsilon}\left[\frac{1}{p_i^h} + \frac{q_i}{p_{i+1}^h} + \ldots + \frac{1}{p_j^h} \prod_{k=1}^{j-1} q_k\right] = \tag{2.16}$$

$$= \frac{H}{\varepsilon}\left[\frac{1}{p_{i+1}^h}\left(\frac{p_{i+1}^h}{p_i^h} + q_i\right) + \frac{q_i q_{i+1}}{p_{i+3}^h}\left(\frac{p_{i+3}^h}{p_{i+2}^h} + q_{i+2}\right) + \ldots\right.$$

$$\left.\ldots + \frac{1}{p_j^h} \prod_{k=i}^{j-2} q_k \left(\frac{1-(-1)^{j-i}}{2} \frac{p_j^h}{p_{j-1}^h} + q_{j-1}\right)\right].$$

If $(j-i)$ is an odd number, by virtue of (2.15) each of the expressions in round brackets in this sum are positive. Their coefficients are also positive and, therefore,

$$A_{ij} > 0. \tag{2.17}$$

The only difference if $(j-i)$ is even is that the quantity in the round brackets in the last term will be negative. But its coefficient will also be negative, and again (2.17) holds.

We now take the first term in the representation of A_{ij} in the form (2.16) to the left-hand side, and group the remaining terms, again in pairs:

$$A_{ij} - \frac{H}{\varepsilon} \frac{1}{p_i^h} = \frac{H}{\varepsilon}\left[\frac{q_i}{p_{i+2}^h}\left(\frac{p_{i+2}^h}{p_{i+1}^h} + q_{i+1}\right) + \ldots\right].$$

The same reasoning leads to the conclusion that

$$A_{ij} - \frac{H}{\varepsilon} \frac{1}{p_i^h} < 0.$$

Combining this inequality with (2.17), we obtain

$$|A_{ij}| < \frac{H}{\varepsilon} \frac{1}{p_i^h} \leqslant \frac{H}{\varepsilon} \frac{1}{p_0}, \tag{2.18}$$

which is consistent with (2.12) when $c_2 = 2$.

3. Suppose q_k varies in sign. Let the natural number m lying between i and $j-1$ be such that all $q_{m+1}, \ldots, q_{j-1}$ are of one sign, and q_m are of the other, or zero. Then when $m+1 \leqslant k \leqslant j-1$, the quantity q_k is described by either (1) or (2) above, which proves the lemma. Thus by virtue of (2.11)

$$|W_{m+1,j}| = |q_{m+1} \cdots q_{j-1}| \leqslant \bar{p} / p_0.$$

Hence, from (1.23), we obtain

$$|W_{ij}| = |W_{im} q_m W_{m+1,j}| \leq |W_{im}| \frac{\bar{p}}{p_0} |q_m|. \tag{2.19}$$

If $q_m = 0$, then $W_{ij} = 0$ and the bound (2.11) is obvious. Thus, let $q_m \neq 0$ and $q_m q_{m+1} < 0$.

Consider the function

$$q(x) = \left[1 - \frac{Hr(x)}{2\varepsilon p(x - H/2)}\right]\left[1 + \frac{Hr(x)}{2\varepsilon p(x + H/2)}\right]^{-1}. \tag{2.20}$$

Bearing (4) and (1.11) in mind, we conclude that $q_m = q(x_m)$ just as $q_{m+1} = q(x_{m+1})$. Since, by hypothesis, q_m and q_{m+1} have different signs, the function $q(x)$, being continuous, vanishes at a point $\xi \in (x_m, x_{m+1})$. Thus we find that

$$\frac{H}{\varepsilon} = 2\frac{p(\xi - H/2)}{r(\xi)} \le \frac{2\bar{p}}{r_0} \tag{2.21}$$

and

$$q_m = q(\xi) + (x_m - \xi)q'(\eta) = (x_m - \xi)q'(\eta), \qquad \eta \in (x_m, \xi). \tag{2.22}$$

We now obtain bounds for $q'(x)$. It follows from (2.20) that

$$q'(x) = -\frac{H}{2\varepsilon}\left[1 + \frac{Hr(x)}{2\varepsilon p(x + H/2)}\right]^{-1}\left[\left(\frac{r(x)}{p(x - H/2)}\right)' + \right.$$

$$\left. + \left(\frac{r(x)}{p(x + H/2)}\right)' q(x)\right].$$

For the first two factors we have

$$\frac{H}{2\varepsilon}\left[1 + \frac{Hr(x)}{2\varepsilon p(x + H/2)}\right]^{-1} \le \frac{\bar{p}}{r_0}.$$

Then, since (2), (2.1) and (2.9) imply that

$$\left|\left[\frac{r(x)}{p(x \pm H/2)}\right]'\right| \le \frac{c_1(\bar{p} + \bar{r})}{p_0^2},$$

and from (2.13) and (2.15) we have $|q(x)| \le \bar{p}/p_0$, it follows that

$$|q'(x)| \le \frac{\bar{p}}{r_0}\left[\frac{c_1(\bar{p} + \bar{r})}{p_0^2}\left(1 + \frac{\bar{p}}{p_0}\right)\right] \equiv c_3 \tag{2.23}$$

and q_m of (2.22) satisfies the inequality

$$|q_m| \le c_3 H. \tag{2.24}$$

Substituting this bound into (2.19), and noting that, by (5),

$$H = \frac{1 - \delta}{N - n} \le \frac{1}{(1 - 1/B)N},$$

we will have

$$|W_{ij}| \le |W_{im}| \frac{\bar{p}}{p_0} \frac{c_3}{(1 - 1/B)N}. \tag{2.25}$$

Let the number N_2, which occurs in the lemma, satisfy the inequality

$$N_2 \ge \frac{\bar{p}c_3}{p_0(1 - 1/B)}. \tag{2.26}$$

Then from (2.25) and (2.10) we find that

$$|W_{ij}| \le |W_{im}|. \tag{2.27}$$

This completes the first stage of finding a bound for $|W_{ij}|$. In the second stage, by picking out the next change of sign of q_k and repeating the above argument, we obtain the bound

$$|W_{im}| \le |W_{im_1}|, \tag{2.28}$$

where the number $m_1 < m-1$ is such that $q_{m_1+1}, \ldots, q_{m-1}$ are of one sign, and q_{m_1} are of the other, or zero. Hence from (2.27) we have

$$|W_{ij}| \le |W_{im_1}|.$$

This process can be continued until there is no change of sign in the remaining q_k which form W_{im_l} on the right-hand side of a bound like (2.28). Then these q_k satisfy one of the conditions (1) and (2), which completes the proof of (2.11).

We now turn to A_{ij}. From (1.24) and (1.23),

$$A_{ij} = A_{im} + W_{i,m+1}A_{m+1,j} = A_{im} + W_{im}q_m A_{m+1,j}.$$

By virtue of the choice of m, allowing for (2.14) and (2.18), $A_{m+1,j}$ satisfies inequality (2.12) with $c_2 = 2$. For W_{im} we have the bound (2.11), and for q_m we have the bound (2.24). Thus

$$|A_{ij}| \le |A_{im}| + \frac{\bar{p}}{p_0 r_0}\left(1+\frac{Hr_0}{2\varepsilon\bar{p}}\right)2\frac{\bar{p}c_3}{p_0}H.$$

Similarly,

$$|A_{im}| \le |A_{im_1}| + \frac{\bar{p}}{p_0 r_0}\left(1+\frac{Hr_0}{2\varepsilon\bar{p}}\right)2\frac{\bar{p}c_3}{p_0}H,$$

and substituting this into the previous inequality we obtain

$$|A_{ij}| \le |A_{im_1}| + \frac{\bar{p}}{p_0 r_0}\left(1+\frac{Hr_0}{2\varepsilon\bar{p}}\right)4\frac{\bar{p}c_3}{p_0}H,$$

and so on, until we obtain the inequality

$$|A_{ij}| \le |A_{im_l}| + \frac{\bar{p}}{p_0 r_0}\left(1+\frac{Hr_0}{2\varepsilon\bar{p}}\right)2\frac{\bar{p}c_3}{p_0}(l+1)H.$$

Since the total number of stages $l+1$ is not greater than $N-n$, and by (2.14) and (2.18) $|A_{im_l}|$ satisfies (2.12) with $c_2 = 2$, we find that

$$|A_{ij}| \le \frac{\bar{p}}{p_0 c_0}\left(1+\frac{Hr_0}{2\varepsilon\bar{p}}\right)2\left(1+\frac{\bar{p}c_3}{p_0}\right) \equiv \frac{\bar{p}}{p_0 c_0}\left(1+\frac{Hr_0}{2\varepsilon p_0}\right)c_2. \tag{2.29}$$

This establishes estimate (2.12) in the general case, which completes the proof of the lemma.

Lemma 3. If the conditions of Lemmas 1 and 2 are satisfied and, in addition, $N \ge \max\{N_3, N_4\}$, where

$$N_3 \ge \left(\frac{4\bar{p}^2\bar{r}c_2}{p_0^2 r_0}\right)^{\bar{p}/(Cr_0)}, \qquad N_4 \ge \frac{\bar{r}c}{2p_0 A(1-1/B)}\ln N_4, \tag{2.30}$$

then

$$\max_{i\ge n}|\alpha_i - \alpha_n| \le \frac{\bar{p}c_2}{p_0 r_0}w_n, \tag{2.31}$$

and

$$\alpha_N \ge c_4 = \frac{p_0}{2\bar{p}\bar{r}}\left[1-\exp\left(-\frac{A\bar{r}}{p_0}\right)\right]. \tag{2.32}$$

Proof. We will prove (2.31). From (1.22) and (1.12) we find that

$$\alpha_i - \alpha_n = A_{n+1,i}w_{n+1} = A_{n+1,i}w_n q_n. \tag{2.33}$$

By virtue of (1.11) and (5), the last factor on the right-hand side of (2.33) has the form

$$q_n = \left(1 - \frac{hr_n^h}{2\varepsilon p_n^h}\right)\left(1 + \frac{Hr_n^h}{2\varepsilon p_{n+1}^h}\right)^{-1}.$$

Condition (2.2) of Lemma 1 ensures that inequality (2.7) holds and this, in turn, leads to the first factor in q_n being positive. Thus, from (2), (2.1) and (2.4), we have

$$|q_n| = q_n \le \left(1 - \frac{hr_0}{2\varepsilon \bar{p}}\right)\left(1 + \frac{Hr_0}{2\varepsilon \bar{p}}\right)^{-1}.$$

Substituting this bound and the bound (2.12) into (2.33), we obtain

$$|\alpha_i - \alpha_n| \le \frac{\bar{p}c_2}{p_0 r_0} w_n \left(1 - \frac{hr_0}{2\varepsilon \bar{p}}\right), \tag{2.34}$$

whence we obtain (2.31).

Note that, by (2.5) and (2.4), (2.34) implies that:

$$|\alpha_i - \alpha_n| \le \frac{\bar{p}c_2}{p_0 r_0} \bar{q}^n. \tag{2.35}$$

We will now prove (2.32). We have

$$\alpha_N = \alpha_n + (\alpha_N - \alpha_n) \ge \alpha_n - |\alpha_N - \alpha_n|.$$

Using (2.3), (2.35) and (2.8), we find that

$$\alpha_N \ge \frac{p_0}{\bar{p}\bar{r}}(1 - \mathring{q}_n) - \frac{\bar{p}c_2}{p_0 r_0} \bar{q}^n \ge \frac{p_0}{\bar{p}\bar{r}} - \frac{2\bar{p}c_2}{p_0 r_0} \bar{q}^n. \tag{2.36}$$

Since

$$\ln \frac{1+t}{1-t} \ge 2t \quad \text{for} \quad 0 \le t < 1,$$

allowing for (2.4) and (5) we have

$$\bar{q}^n = \exp\left(-n \ln \frac{1}{\bar{q}}\right) \le \exp\left(-n \frac{hr_0}{\varepsilon \bar{p}}\right) = \exp\left(-\frac{\delta r_0}{\varepsilon \bar{p}}\right).$$

Let the grid (5) be such that

$$C\varepsilon \ln N < A. \tag{2.37}$$

Then $\delta = C\varepsilon \ln N$ and, from (2.2) and (2.30), we find that

$$\bar{q}^n \le \exp\left\{-\frac{Cr_0 \ln N}{\bar{p}}\right\} = N^{-Cr_0/\bar{p}} \le N_3^{-Cr_0/\bar{p}} = \frac{p_0^2 r_0}{4\bar{p}^2 \bar{r} c_2}.$$

Hence under the condition (2.37), (2.36) implies (2.32).

But if, instead of (2.37), we have $C\varepsilon \ln N \ge A$, everything is much simpler and bounds for α_N must be found in a different way. Since in that case $\varepsilon \ge A/(C \ln N)$, using (2.30) we have

$$\frac{Hr_i^h}{2\varepsilon p_i^h} \le \frac{H\bar{r}}{2\varepsilon p_0} < \frac{\bar{r} C \ln N}{2A(1 - 1/B) p_0 N} \le \frac{C\bar{r} \ln N_4}{2A(1 - 1/B) p_0 N_4} \le 1$$

and therefore q_k of (1.11) is positive not only for $k = 1, 2, \ldots, n$, which follows from (2.2) and (2.7), but also for $k = n+1, \ldots, N-1$. Hence from (1.12) we see that w_i is positive, and from (1.13) we see that α_i is monotonely increasing for all $i = 1, 2, \ldots, N$. Thus, in particular, $\alpha_N \ge \alpha_n$ and this, together with (2.3), leads to the inequality

$$\alpha_N \ge \frac{p_0}{\bar{p}\bar{r}}(1 - \mathring{q}^n). \tag{2.38}$$

Then, as before, since $hn = A$ we have

$$\mathring{q}^n \le \exp\left(-\frac{nh\bar{r}}{\varepsilon p_0}\right) = \exp\left(-\frac{A\bar{r}}{\varepsilon p_0}\right) \le \exp\left(-\frac{A\bar{r}}{p_0}\right),$$

for $\varepsilon \leqslant 1$. This, together with (2.38), gives (2.32), which proves the lemma.

3. BOUNDS FOR GREEN'S FUNCTION AND CONVERGENCE

We now have everything we need to establish a uniform bound for Green's function with respect to ε.

Theorem 2. Under the conditions of Lemmas 1–3,

$$|G(x_i, \xi_j)| \le c_5, \tag{3.1}$$

where $c_5 = c_5(p, r)$ (cf. (3.3)) and is independent of N and ε.

Proof. We first find a bound for $G(x_i, \xi_j)$ for $j \ge n$, and then for $j < n$.

a. Let $j \ge n$. From (1.25), and Lemmas 2 and 3 we find that

$$|G(x_i, \xi_j)| \le \frac{\bar{p}c_2}{p_0 r_0 c_4}\begin{cases}|\alpha_i|, & i \le j,\ \ j \ge n,\\ \dfrac{\bar{p}}{p_0}|\alpha_j|, & i \ge j \ge n.\end{cases} \tag{3.2}$$

If $i \le n$, by Lemma 1, $\alpha_i \le \alpha_n$, but if $i > n$, then

$$|\alpha_i| = |\alpha_n + (\alpha_i - \alpha_n)| \le \alpha_n + |\alpha_i - \alpha_n|,$$

so that in any case

$$|\alpha_i| \le \alpha_n + |\alpha_i - \alpha_n|.$$

From this and (2.3) and (2.35) it follows that

$$|\alpha_i| \le \bar{p}c_2/(p_0 r_0).$$

Bounds for $|\alpha_j|$ are found in exactly the same way. Together with (3.2), they give the bound

$$|G(x_i, \xi_j)| \le c_5 \equiv \frac{\bar{p}^3 c_2^2}{p_0^3 r_0^2 c_4}, \qquad j \ge n. \tag{3.3}$$

b. We now find bounds for $G(x_i, \xi_j)$ when $j < n$. We will use representations (1.20). We start with the bounds for $(\alpha_N - \alpha_j)$. We have

$$\alpha_N - \alpha_j = (\alpha_N - \alpha_n) + (\alpha_n - \alpha_j).$$

Bounds for the first term on the right-hand side can be found using (2.31), and by virtue of (1.22) the second has the form

$$\alpha_n - \alpha_j = w_{j+1} A_{j+1,n}.$$

Since $j < n$, by Lemma 1, $q_j > 0$ and $A_{j+1,n}$ has a bound of the form (2.14) with h instead of H. Bearing all this in mind, we will have

$$|\alpha_N - \alpha_j| \le \frac{\bar{p}}{p_0 r_0}\left[c_2 w_n + \left(1 + \frac{hr_0}{2\varepsilon\bar{p}}\right) w_{j+1}\right]. \tag{3.4}$$

Then, by Lemma 1, the function w_j is decreasing for $j < n$ and thus $w_n \leqslant w_{j+1}$. Substituting this bound into (3.4) and reducing the accuracy slightly, we finally obtain

$$|\alpha_N - \alpha_j| \le \frac{2\bar{p}c_2}{p_0 r_0}\left(1 + \frac{hr_0}{2\varepsilon\bar{p}}\right) w_{j+1}. \tag{3.5}$$

We will prove that the values $(\alpha_N - \alpha_i)$ in which we are interested have exactly the same bound. In fact if $i < n$, then we need only replace j by i in (3.5). But since $(\alpha_N - \alpha_i)$ occurs in (1.20) only for $i \geqq j$, and w_i is a decreasing function, we can put w_{j+1} instead of w_{i+1} in the new bound for $|\alpha_N - \alpha_i|$.

But if $i > n$, then $(\alpha_N - \alpha_i) = (\alpha_N - \alpha_n) + (\alpha_n - \alpha_i)$ and inequality (2.31) is applicable to both terms, by virtue of which $|\alpha_N - \alpha_i| \le [2\bar{p}c_2/(p_0 r_0)]w_n$. A bound for the right-hand side of this inequality in terms of the right-hand side of (3.4) is obtained similarly.

Finally, we find bounds for α_i and α_j. Since α_i occurs in (1.20) only when $i \le j$, and the function α_i, by Lemma 1, is increasing for $i \le n$, from (2.3) we have

$$\alpha_i \le \alpha_j \le \alpha_n \le \bar{p}/(p_0 r_0). \tag{3.6}$$

Thus, all the preliminary bounds have been obtained. We now substitute (3.6), (3.3) and the analogue of (3.5) for $(\alpha_N - \alpha_i)$ into (1.20). If we then allow for the fact that

$$\left(1+\frac{r_j^h h_{j+1}}{2\varepsilon p_{j+1}^h}\right) \ge \left(1+\frac{h r_0}{2\varepsilon \bar{p}}\right),$$

and that α_N satisfies (2.32), we obtain the bound

$$|G(x_i, \xi_j)| \le \frac{2\bar{p}^2 c_2}{p_0^2 r_0^2 c_4}, \qquad j < n,$$

which, as is easily seen (cf. the definition of c_2 in (2.29), for example), is not worse than (3.3). This proves the theorem.

We have everything we need to prove Theorem 1. Let

$$z_i = u_i^h - u_i.$$

where u_i and u_i^h are the exact solution of problem (1) and the approximate solutions at grid nodes. Then, as usual,

$$L^h z_i = \psi_i, \qquad x_i \in \Omega, \quad z_0 - z_N = 0,$$

where $\psi_i = f_i^h - L^h u_i = \varepsilon[(p^h u_{\bar{x}})_{\hat{x}} - (pu')']_i + r_i[u_{\mathring{x}} - u']_i$ is the approximation error for problem (6).

By (1.4)

$$z(x_i) = (G(x_i, \xi_j), \psi(\xi_j)).$$

Hence, from Theorem 2, we obtain the bound

$$\|z\|_{L_\infty^h(\Omega)} = \max_{x_i \in \Omega} |z_i| \le c_5 \sum_{x_i \in \Omega} |\psi_i| \hbar_i = c_5 \|\psi\|_{L_1^h(\Omega)}. \tag{3.7}$$

Now repeating the arguments used in [3] to prove Theorem 4 in a simpler form, we can see that

$$\|\psi\|_{L_1^h(\Omega)} = O(N^{-2} \ln^2 N)$$

uniformly with respect to ε. This bound, together with (3.7), completes the proof of Theorem 1.

Remarks. 2. As we can see from the analysis of Eq. (1) with constant coefficients and a zero right-hand side, the factor $\ln^2 N$ cannot be omitted from the estimate of accuracy of the difference scheme (6) on the grid (5).

3. From the same example we see that, in the best case, condition (7) can be relaxed without affecting the result only up to $C \ge 2p(0)/r(0)$.

4. If the grid is uniform, then $u_{\mathring{x}} = 0.5(u_x + u_{\bar{x}})$. This is not true on a non-uniform grid. The analysis of the above example shows that replacing $u_{\mathring{x}}^h$ in (6) by $0.5(u_x^h + u_{\bar{x}}^h)$ destroys the uniform convergence with respect to ε, even though this replacement actually only affects the equation with $i = n$. Replacing $u_{\mathring{x},n}^h$ in (6) by $(hu_x^h + Hu_{\bar{x}}^h)_n/(2\hbar_n)$ has the same effect. It is easy to see that the last expression approximates $u'(x_n)$ on a non-uniform grid with error $O(H^2 + h^2)$.

5. If instead of (2), the coefficient $r(x)$ of Eq. (1) satisfies the condition $r(x) \le -r_0 < 0$, then Theorem 1 remains true when the grid Ω of (5) is replaced by a grid Ω' which clusters, like Ω, but at the right-hand end of the interval [0, 1], and the parameter C is replaced by $C > 2p(1)/|r(1)|$.

6. If the problem for the conjugate equation to (1) is approximated with the help of the adjoint operator (1.5) to L^h of (6) in the sense (1.3), Theorem 1 remains true for the grid Ω' of Remark 5.

4. NUMERICAL RESULTS

We will now give results to illustrate the accuracy of the scheme. It is easy to see that the function

$$u(x) = \frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} + 2x\cos\frac{\pi x}{2}$$

is a solution of the problem

Table 1

N	ε				
	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}
8	0.00211 4.0	0.08554 3.2	0.09580 3.3	0.09596 3.2	0.09596 3.2
16	0.00053 4.0	0.02672 3.1	0.02946 3.2	0.02955 3.2	0.02955 3.2
32	0.00013 4.0	0.00850 3.1	0.00922 3.2	0.00929 3.2	0.00929 3.2
64	0.00003 4.0	0.00271 3.1	0.00287 3.3	0.00292 3.2	0.00292 3.2
128	0.00001 4.0	0.00086 3.2	0.00088 3.2	0.00091 3.2	0.00091 3.2
256	0.00000	0.00027	0.00027	0.00028	0.00028

Table 2

i	x_i	z_i	i	x_i	z_i
0	0.00000	0.00000	11	0.10053	−0.01210
1	0.00005	−0.01608	12	0.20047	−0.00562
2	0.00011	−0.02010	13	0.30041	−0.00757
3	0.00017	−0.01972	14	0.40035	−0.00161
4	0.00023	−0.01810	15	0.50029	−0.00418
5	0.00029	−0.01647	16	0.60023	0.00092
6	0.00035	−0.01516	17	0.70017	−0.00253
7	0.00041	−0.01422	18	0.80011	0.00153
8	0.00047	−0.01359	19	0.90005	−0.00295
9	0.00053	−0.01318	20	1.00000	0.00000
10	0.00059	−0.01292			

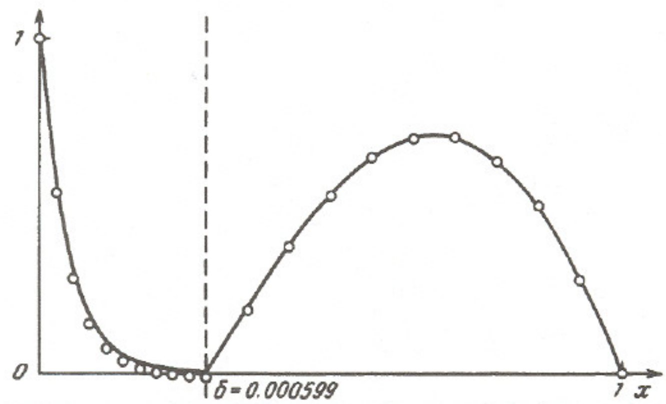

Fig. 1.

$$\varepsilon u'' + u' = -f(x), \quad 0 < x < 1, \qquad u(0) = 1, \quad u(1) = 0,$$

with right-hand side

$$f(x) = \left(\frac{\varepsilon\pi^2 x}{2} - 2\right)\cos\frac{\pi x}{2} + \pi(2\varepsilon + x)\sin\frac{\pi x}{2}.$$

This problem was solved on the grid (5) with $B = 1/2$, that is, with the same number of nodes in and outside the "boundary layer". The parameter A was taken equal to 1/2, and $C = 2$. Table 1 gives the values of the L_∞^h-norm of the error of the solution for different ε and N and the rate at which the error decreases when the number of nodes is doubled (the second number in each row). A row by row analysis of Table 1 for each N shows that the error stabilizes as $\varepsilon \to 0$, reflecting the uniform convergence. It is clear from the columns of Table 1 that the rate of convergence is not less than the predicted value, as even when $N = 128$ we obtain $\{(N^{-1} \ln N)/[(2N)^{-1} \ln(2N)]\}^2 \approx 3.06$.

Table 2 gives the pointwise error of the solution for $\varepsilon = 10^{-4}$ and $N = 20$. Clearly, on a fine mesh the error in the "boundary layer" changes smoothly without oscillating, whereas on a coarse grid small oscillations of the error are observed outside the "boundary layer", reflecting non-monotonicity of the scheme. The solid curve in Fig. 1 depicts the exact solution for $\varepsilon = 10^{-4}$, and the circles the approximate solution for $N = 20$. The scale in the boundary layer of width $\delta = 2\varepsilon \ln N$ has been increased for clarity.

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