Uniform with respect to a Small Parameter Convergence of Difference Schemes for a Convection-Diffusion Problem *

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Abstract

For a steady-state one-dimensional convection-diffusion equation with a small parameter monotone difference schemes are considered on Shishkin grid and Bakhvalov grid condensing in the boundary layer. Their uniform with respect to a small parameter accuracy in L^h_∞-norm is established to be $O(N^{-2}\ln^2 N)$ and $O(N^{-2})$ respectively, where N is the gridpoints number, the smoothness restriction for Bakhvalov grid being replaced by a weaker condition.

1 Introduction

This paper is concerned with ε-uniform numerical methods for the model boundary value problem

$$Lu \equiv -\varepsilon(p(x)u')' - (r(x)u)' = f(x), \quad 0 < x < 1, \quad u(0) = g_0, \quad u(1) = g_1, \tag{1}$$

where
$$p(x) \geq p_0 = \text{const} > 0, \quad r(x) \geq r_0 = \text{const} > 0, \tag{2}$$

and $\varepsilon \in (0, 1]$ is a small parameter.

It is well known [1, 2] that for small ε at $x = 0$ the solution has a boundary layer and as a result the accuracy of classical numerical methods depends on ε as well as on the gridpoints number N. One of the approaches in constructing ε-uniform numerical methods is to use classical discretizations of differential equations on grids condencing in boundary layers. N. S. Bakhvalov [3] was the first to use the approach

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(for second order differential equations without first derivative terms). The grid from [3] for problem (1),(2) is as follows

$$x_i = x(i/N), \quad i = 0, 1, \ldots, N, \tag{3}$$

where $x(t)$ is a continuous function defined by

$$x(t) = \begin{cases} \begin{cases} \varepsilon C \ln\left[b/(b-t)\right] & \text{for} \quad t \in [0, \theta] \\ 1 - d(1-t) & \text{for} \quad t \in [\theta, 1] \end{cases} & \text{if} \quad \varepsilon \le \varepsilon_0 \\ \qquad\qquad t & \text{otherwise} \end{cases} \tag{4}$$
$$d = d(\theta) = \left(1 - \varepsilon C \ln\left[b/(b-\theta)\right]\right)/(1-\theta),$$

with constants C, $0 < \theta < b < 1$, $\varepsilon_0 \le b/C$. Note that the grid from [3] for problems like (1),(2) was considered in [4] and [5], ε-uniform accuracy being obtained to be $O(N^{-1})$ and $O(N^{-2})$ respectively. In the mentioned papers grid (3),(4) is assumed to be smooth, i.e. the function $x(t)$ is continuously differentiable and $\theta = \bar{\theta}$ determined from the nonlinear equation

$$\bar{\theta} = b - \varepsilon C/d(\bar{\theta}) \tag{5}$$

can be computed approximately by (using) the following iterations [3]

$$\theta^{(0)} = 0, \quad \theta^{(k)} = b - \varepsilon C/d(\theta^{(k-1)}), \quad \lim_{k\to\infty} \theta^{(k)} = \bar{\theta}, \quad 0 = \theta^{(0)} < \theta^{(1)} < \ldots < \bar{\theta}. \tag{6}$$

Note that the impossibility of solving a nonliniar equation exactly when constructing the grid can be considered a certain drawback ([2]). In this paper the grid smothness restriction implying (5) is replaced by the following weaker condition

$$b - \varepsilon \bar{c} < \theta < b - \varepsilon c_0, \tag{7}$$

where $0 < c_0 < \bar{c}$ are constants. Here the right hand inequality implies that $\max_i h_i = O(N^{-1})$ for the grid (3),(4), and the left hand inequality provides approximation and convergence. It should be pointed out that the choice $\theta = \bar{\theta}$ is a particular case of (7) as well as $\theta = \theta^{(1)} = b - \varepsilon C$, which is the result of the first iteration (6), and both the choices generates the grids satisfiing the reasonable condition $h_i \le h_{i+1}$ (which is provided by $\theta \le \bar{\theta}$).

G. I. Shishkin [6] suggested another way of condensing grids in boundary layers, in particular for problem (1),(2) the grid from [6] is as follows

$$\Omega = \left\{ x_i \mid x_i = \begin{cases} ih & \text{if} \quad i = 1, 2, \ldots, n, \\ x_n + (i-n)H & \text{if} \quad i = n+1, n+2, \ldots, N-1, \end{cases} \right. \tag{8}$$
$$\left. h = \delta/n, \quad H = (1-\delta)/(N-n), \quad n/N = b, \quad \delta = \min(\varepsilon C \ln N, A) \right\}$$

with constants A, $b \in (0,1)$ and C, and the results from [6] lead to ε-uniform error estimate $O(N^{-1} \ln^2 N)$. In recent papers [7, 8, 9, 10, 11] (see also [1, 2]) on grid (8) other schemes for problems like (1),(2) are studied, ε-uniform accuracy being obtained $O(N^{-2}\ln^2 N)$.

The subject of this paper is the following family of three point difference schemes

$$(L^h u^h)_i \equiv -\left(\varepsilon p_i^h u_{\bar{x},i}^h + \sigma_i r_i^h u_i^h + (1-\sigma_i) r_{i-1}^h u_{i-1}^h\right)_{\widehat{x},i} = f_i^h, \quad i = 1, 2, \dots, N-1,$$
$$u_0^h = g_0, \quad u_N^h = g_1 \tag{9}$$

with the variable weight $$\sigma_i \in [1/2,\ 1], \tag{10}$$

where $$p_i^h = p(x_{i-1/2}), \quad r_i^h = r(x_i), \tag{11}$$

$$f_i^h = \left(f\left(x_{i-1} + h_i\sigma_i\right) + f\left(x_i + h_{i+1}\sigma_{i+1}\right)\right)/\, 2. \tag{12}$$

Here and furthermore the usual notation [12] is used

$$v_{\bar{x},i} = \frac{v_i - v_{i-1}}{h_i}, \quad v_{x,i} = \frac{v_{i+1} - v_i}{h_{i+1}}, \quad v_{\overset{\circ}{x},i} = \frac{v_{i+1} - v_{i-1}}{h_i + h_{i+1}}, \quad h_i = x_i - x_{i-1},$$

as well as the particular notation $$v_{\widehat{x},i} = \frac{v_{i+1} - v_i}{\hbar_{\sigma,i}}, \quad \hbar_{\sigma,i} = h_i(1-\sigma_i) + h_{i+1}\sigma_{i+1}.$$

It is easy to see that setting $\sigma_i \equiv 1/2$ turns scheme (9) into the central difference scheme, and on uniform grids setting $\sigma_i \equiv 1$ turns (9) into the upwind scheme. The wight σ_i is chosen so that scheme (9) have advantages of both the schemes and be free of their drawbacks:

$$1 - \frac{\varepsilon p_i^h}{h_i r_{i-1}^h} \le \sigma_i \le \frac{1}{2} + \mathrm{const}\,\frac{h_i}{\varepsilon}. \tag{13}$$

Here the left hand inequlity provides the maximum principle and the right hand inequality provides the second order approximation on the uniform grid (if σ_i is a smooth grid function).

Conditions (10),(13) leave a certain freedom in choosing the weight σ_i. The following choices seem the most interesting

$$\text{A)}\quad \sigma_i = \begin{cases} \dfrac{1}{2} & \text{if} \quad 1 - \dfrac{\varepsilon p_i^h}{h_i r_{i-1}^h} \le \dfrac{1}{2} \\ 1 & \text{otherwise} \end{cases} \quad ; \qquad \text{B)}\quad \sigma_i = \max\left\{\frac{1}{2},\, 1 - \frac{\varepsilon p_i^h}{h_i r_{i-1}^h}\right\}. \tag{14}$$

It should be noted that similar schemes for a nondivergent problem like (1),(2) were considered in [10, 11] on Shishkin piecewise-uniform grid, ε-uniform accuracy being obtained $O(N^{-2}\ln^2 N)$.

The main result of this paper is ε-uniform convergence of schemes (9)-(13) on condensing grids (3),(4),(7) and (8) (and their nonuniform generalizations), ε-uniform accuracy being $O(N^{-2})$ and $O(N^{-2}\ln^2 N)$ respectively. In conclusion some numerical results are presented.

2 A Priori Estimates and Maximum Principle

Let $\bar{\omega} = \{\, x_i \,|\, 0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1 \,\}$ be an arbitrary nonuniform grid on $[0, 1]$ and for any grid function on $\bar{\omega}$ the norm be defined by

$$||v||_{L^h_\infty} = \max_i |v_i|.$$

Three point difference scheme (9) under consideration can be rewritten as follows

$$\left(L^h v\right)_i = -\left(A^h v\right)_{\widehat{x},i} = f_i^h, \tag{15}$$

where the difference operator A^h is defined by

$$\left(A^h v\right)_i \equiv \varepsilon p_i^h v_{\bar{x},i} + \sigma_i r_i^h v_i + (1 - \sigma_i) r_{i-1}^h v_{i-1}. \tag{16}$$

Note that $A^h v$ approximates the difference expression $Av \equiv \varepsilon p(x) v' + r(x) v$, and $-(Av)' = Lv$.

Theorem 1 *Let the coefficient functions $p(x)$ and $r(x)$ of eq. (1) satisfy (2) and v_i be the solution of difference equation (15),(16),(10),(11),(13) with zero boundary conditions $v_0 = v_N = 0$. Then on an arbitrary nonuniform grid $\bar{\omega}$ the following estimate holds*

$$||v||_{L^h_\infty} \leq \frac{2}{r_0} \max_i \left| \sum_{j=i}^{N-1} f_j^h \hbar_{\sigma,j} \right|. \tag{17}$$

Proof: Multiplying each of equations (15) by $\hbar_{\sigma,i}$ it is easy to see that

$$\left(A^h v\right)_i = a + \sum_{j=i}^{N-1} f_j^h \hbar_{\sigma,j}, \quad i = 1, 2, \ldots, N$$

with a constant a. Since this difference equation is liniar, its solution is represented as $v_i = aV_i - W_i$, where V_i and W_i are the solutions of the following difference problems

$$\left(A^h V\right)_i = 1, \quad i = 1, 2, \ldots, N, \quad V_0 = 0, \tag{18}$$

$$\left(A^h W\right)_i = \sum_{j=i}^{N-1} f_j^h \hbar_{\sigma,j}, \quad i = 1, 2, \ldots, N, \quad W_0 = 0, \tag{19}$$

and the boundary condition $v_N = 0$ implies $a = W_N / V_N$, which leads to

$$v_i = \frac{W_N V_i}{V_N} - W_i. \tag{20}$$

Since from (16) $\quad \left(A^h v\right)_i = -\left(\varepsilon \dfrac{p_i^h}{h_i} - (1 - \sigma_i) r_{i-1}^h\right) v_{i-1} + \left(\varepsilon \dfrac{p_i^h}{h_i} + \sigma_i r_i^h\right) v_i,$

(18) and (19) are algebraic equations systems with a two diagonal matrix, and from (10),(13) the matrix elements a_{ij} satisfy

$$a_{ii} > 0,\ a_{i,i-1} \leq 0, \quad a_{ii} - |a_{i,i-1}| \geq \sigma_i r_i^h + (1-\sigma_i) r_{i-1}^h \geq r_0.$$

Then [12, 13] the matrix is an M-matrix, and for problems (18) and (19) the maximum principle holds and gives the bounds

$$0 < V_i \leq 1/r_0, \quad |W_i| \leq V_i \max_i \left| \sum_{j=i}^{N-1} f_j^h \hbar_{\sigma,j} \right|, \quad i = 1, 2, \ldots, N$$

leading from (20) to (17), which completes the proof.

Theorem 2 *The grid operator L^h defined by (9) with $g_0 = g_N = 0$, (10),(11),(13) satisfies the maximum principle [12, p. 60].*

Proof: It is sufficient to establish that the three diagonal matrix of the difference operator $\tilde{L}^h$, defined by $\left(\tilde{L}^h v\right)_i \equiv \left(L^h v\right)_i \hbar_{\sigma,i}$, is an M-matrix, see [13].

3 Approximation and Convergence

Theorem 3 *Let $u(x)$ be the solution of (1),(2) with sufficiently smooth $p(x)$, $r(x)$ and $f(x)$, and u_i^h be the solution of (9)-(13). Then on an arbitrary nonuniform grid $\bar{\omega}$ the following ε-uniform estimate holds*

$$||u_i^h - u(x_i)||_{L_\infty^h} = O\left(\max_i \left\{ h_i^2 + \min\left[\left(\frac{h_i}{\varepsilon}\right)^2, 1\right] \exp\left(-\frac{\gamma x_{i-1}}{\varepsilon}\right)\right\}\right) \tag{21}$$

where γ is an arbitrary constant satisfiing $\gamma = \mathrm{const} < r(0)/p(0)$.

Proof: Let $z_i \equiv u_i^h - u(x_i)$ be the error. Then

$$(L^h z)_i \equiv -\left(A^h z\right)_{\widehat{x},i} = \psi_i, \quad i = 1, \ldots, N-1, \quad z_0 = z_N = 0, \tag{22}$$

where
$$\psi_i \equiv f_i^h - \left(L^h u\right)_i = \left(A^h u\right)_{\widehat{x},i} + f_i^h.$$

is the truncation error. Setting $\bar{x}_i = x_{i-1} + h_i \sigma_i$ and integrating eq. (1) over $[\bar{x}_i, \bar{x}_{i+1}]$ leads to

$$\left[(Au)(\bar{x}_i)\right]_{\widehat{x},i} + \frac{1}{\hbar_{\sigma,i}} \int_{\bar{x}_i}^{\bar{x}_{i+1}} f(x)\,dx = 0,$$

so the truncation error ψ_i can be written as follows

$$\psi_i = \eta_{\widehat{x},i} + \mathring{\psi}_i, \quad \eta_i \equiv \left(A^h u\right)_i - (Au)(\bar{x}_i), \quad \mathring{\psi}_i = f_i^h - \frac{1}{\hbar_{\sigma,i}} \int_{\bar{x}_i}^{\bar{x}_{i+1}} f(x)\,dx.$$

Putting this reprsentation of ψ_i into (22), using theorem 1 and noting that from (12) $\max_i \left| \sum_{j=i}^{N-1} \overset{\circ}{\psi}_j \hbar_{\sigma,j} \right| \leq || \overset{\circ}{\psi} ||_{L^h_\infty} = O\left(\hbar^2_{\sigma,i}\right)$ ε-uniformly, we obtain the ε-uniform estimate

$$||z||_{L^h_\infty} \leq \frac{2}{r_0}\left(2\,||\eta||_{L^h_\infty} + ||\,\overset{\circ}{\psi}\,||_{L^h_\infty}\right) = \frac{4}{r_0}\,||\eta||_{L^h_\infty} + O\left(\max_i h_i^2\right). \tag{23}$$

Here η_i is estimated using the decomposition of $u(x)$ into a smooth component and a singular component [6, p. 197] and Taylor series expansions, which completes the proof.

Theorem 4 *Let $u(x)$ be the solution of (1),(2) with sufficiently smooth $p(x)$, $r(x)$ and $f(x)$, and u_i^h be the solution of (9)-(13). Moreover, let the grid points be $x_i = x(t_i)$ with the function $x(t)$ defined by*

$$a)\ (4),(7) \quad or \quad b)\ x(t) = \begin{cases} \dfrac{\delta}{b}t, & t \in [0,b] \\ \delta + \dfrac{1-\delta}{1-b}(t-b), & t \in [b,1] \end{cases} \qquad \delta = \min(\varepsilon C \ln N, A),$$

where A, $b \in (0,1)$, C are constants; and let t_i be the points of a grid $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = 1$ satisfying the only restriction $t_i - t_{i-1} = O(N^{-1})$. Then, provided that on the grid parameter C from (4) and b) satisfies the condition $C > 2p(0)/r(0)$, the following ε-uniform estimates hold

$$a)\ ||u_i^h - u(x_i)||_{L^h_\infty} = O(N^{-2}); \qquad b)\ ||u_i^h - u(x_i)||_{L^h_\infty} = O(N^{-2}\ln^2 N).$$

Proof: The estimates that are to be proved are derived from theorem 3 due to detailed analysis of the right hand terms in (21) for the grids under consideration.
Remark: If $t_i = i/N$, $i = 0, 1, \ldots, N$, the grids a) and b) from theorem 4 turn into (4),(8) [3] and (8) [6], i.e. the grids a) and b) are nonuniform generalizations of the Bakhvalov and Shishkin grids.

4 Numerical results

To illustrate the accuracy of the schemes under consideration we present some numerical results. As it is easy to see $u(x) = \dfrac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}} + 2x\cos\dfrac{\pi x}{2}$ is the solution of the problem

$$\varepsilon u'' + u' = \left(2 - \frac{\varepsilon\pi^2 x}{2}\right)\cos\frac{\pi x}{2} - \pi(2\varepsilon + x)\sin\frac{\pi x}{2},\ 0 < x < 1,\ u(0) = 1,\ u(1) = 0,$$

which was solved by scheme (9),(11),(12),(14)B) on Bakhvalov grid (3),(4),(5), $C = 2.3$, $b = 0.5$, $\varepsilon_0 = bd/C$ and Shishkin grid (8), $C = 2.3$, $b = A = 0.5$. In tables 1

Table 1: **Error for Bakhvalov grid**

N	$\varepsilon = 2^{0}$	$\varepsilon = 2^{-3}$	$\varepsilon = 2^{-6}$	$\varepsilon = 2^{-9}$	$\varepsilon = 2^{-12}$	$\varepsilon = 2^{-15}$	$\varepsilon = 2^{-18}$	$\max_{\varepsilon}$
16	0.00067	0.00617	0.00327	0.00720	0.00802	0.00817	0.00819	0.00819
	4.0	4.0	3.3	4.0	4.0	4.0	4.0	3.6
32	0.00017	0.00152	0.00098	0.00179	0.00200	0.00204	0.00204	0.00227
	4.0	4.0	1.7	4.1	4.0	4.0	4.0	3.9
64	0.00004	0.00038	0.00059	0.00044	0.00050	0.00051	0.00051	0.00059
	4.0	4.0	4.0	4.3	4.0	4.0	4.0	4.0
128	0.00001	0.00009	0.00015	0.00010	0.00013	0.00013	0.00013	0.00015
	4.0	4.0	4.0	6.1	4.0	4.0	4.0	4.0
256	0.00000	0.00002	0.00004	0.00002	0.00003	0.00003	0.00003	0.00004
	4.0	4.0	4.0	1.9	4.1	4.0	4.0	4.0
512	0.00000	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001

and 2 respectively one can see the error in L^h_∞ - norm for different ε and N and how many times the error decreases as N increases twice. In the last column there are the values of

$$\max_{\varepsilon=2^{-i},\, i=0,1,2,\ldots,21} ||u^h_i - u(x_i)||_{L^h_\infty} \tag{24}$$

for different N. Analysing the lines of the tables one can see that the error stabilizes as $\varepsilon \to 0$ (i.e. the method is ε-uniform). The column analysis shows that though for certain combinations of ε and N the error decreases only 1.7 times as the gridpoints number increases twice, for large enough N the values of (24) from the last column decreases 4 times (table 1) and 3.2 times (table. 2), i.e. the numerical convergence order agrees with the theoretical estimates established.

Table 2: **Error for Shishkin grid**

N	$\varepsilon = 2^{0}$	$\varepsilon = 2^{-3}$	$\varepsilon = 2^{-6}$	$\varepsilon = 2^{-9}$	$\varepsilon = 2^{-12}$	$\varepsilon = 2^{-15}$	$\varepsilon = 2^{-18}$	$\max_{\varepsilon}$
16	0.00067	0.00868	0.01833	0.01621	0.01595	0.01592	0.01591	0.02246
	4.0	4.1	2.4	2.4	2.4	2.4	2.4	2.6
32	0.00017	0.00213	0.00766	0.00662	0.00653	0.00652	0.00652	0.00864
	4.0	4.0	2.6	2.7	2.7	2.7	2.7	2.9
64	0.00004	0.00053	0.00295	0.00248	0.00245	0.00245	0.00245	0.00295
	4.0	4.0	3.0	2.9	2.9	2.9	2.9	3.0
128	0.00001	0.00013	0.00099	0.00087	0.00085	0.00085	0.00085	0.00099
	4.0	4.0	3.1	2.9	3.0	3.0	3.0	3.1
256	0.00000	0.00003	0.00032	0.00029	0.00028	0.00028	0.00028	0.00032
	4.0	4.0	3.2	3.0	3.1	3.1	3.1	3.2
512	0.00000	0.00001	0.00010	0.00010	0.00009	0.00009	0.00009	0.00010

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